Chambers's Educational Course.

Now issued, strongly bound in Cloth:

Directories.

	s.	d.
Infant Treatment	1	3
——— **Education**	2	0

English.

	s.	d.
Elementary Lessons, in 14 Sheets, 23 by 18 inches......each	0	2
First Book of Reading, *sewed*	0	1½
Second Book of Reading, *do.*	0	3
Simple Lessons in Reading	0	8
Rudiments of Knowledge	0	8
Lesson Book of Common Things	0	8
Moral Class-Book	1	6
Composition, Introduction to	0	6
Grammar, Introduction to	0	8
Grammar and Composition	1	6
Etymology, Exercises on	2	0
Elocution, Principles of	2	6
English Language and Literature	2	6

Writing and Drawing.

	s.	d.
Writing: 15 Post Copy-Books, each	0	6
——— **13 Fine Post Copy-Books,** "	0	6
——— **10 Foolscap Copy-Books,** "	0	3
First Drawing Book	1	0
Second Drawing Book	1	0
Drawing Books—A series of Progressive Lessons in Drawing and Perspective, in 18 Books, *swd.*..each	1	6
Mechanical Drawing, 3 Bks. *swd.* "	1	6
Architectural Drawing, 3 Bks. *swd.* "	2	0
Isometrical Drawing, 2 Bks. *swd.* "	2	0
Ornamental Drawing, in 3 sheets, each 13 by 9 inches......each	0	3
Styles of Gothic Architecture, in a sheet 25 by 20 inches	1	0
Orders of Roman Architecture, in a sheet 25 by 20 inches	1	0
Condensing Beam Steam-Engine, &c. in a sheet 25 by 20 inches	1	0
Working Drawings and Details of Steam-Engines, 4to, *sewed*	1	0
Illustrations of Machine and Mill Gearing, 4to, *sewed*	1	0

Geography.

Geographical Primer......

England, Geographical Te

Scotland..........do.........

Scripture Geography......

Geography, General Trea

Geography.

Large School-Room Maps—

(*5 feet 2 inches long, by 4 feet 6 inches broad.*)

		s.	d.
England	mounted,	14	0
Scotland	" "	14	0
Ireland	" "	14	0
Europe	" "	14	0
Palestine	" "	14	0
Asia	" "	14	0
Africa	" "	14	0
North America	" "	14	0
South America	" "	14	0
The Hemispheres	" "	21	0

These may be had varnished, at 2s. 6d. extra.

These 10 Maps in Wooden Case (10 Maps, £7, 7s.; Case, £1, 10s.)......£8 17 0

	s.	d.
Primer Atlas; containing 9 quarto Maps, coloured	2	6
Minor Atlas; containing 15 quarto Maps, coloured	5	0
School Atlas of Modern & Ancient Geography; containing 34 quarto Maps, coloured; with Index	10	6

Small School and Family Maps; prepared by W. & A. K. Johnston.

(*2 feet 9 inches long, by 2 feet 3 inches broad.*)

Eastern Hemisphere, Western Hemisphere, England, Scotland, Ireland, Europe, Asia, Africa, America, Canaan and Palestine—[*These 10 Maps in Wood Case,* £4, 4s.; *or in Wooden Stand,* £5, 5s.]—United States and Canada, Chart of the World. Each Map, mounted......5 0

These Maps may also be had varnished, at 1s. extra.

History.

	s.	d.
Ancient History	3	0
Medieval History	3	6
Modern History	4	6
Greece, History of Ancient	3	0
Rome, History of	2	6
British Empire, History of the	2	0
Biography, Exemplary & Instructive	2	0

Arithmetic & Mathematics.

	s.	d.
Introduction to Arithmetic	1	0
Arithmetic, Advanced *New Edition,*	2	0
Key to Arithmetic, ...*do.*	2	0
Book-Keeping by S. and D. Entry	1	6
...by Single Entry	1	0
...oks for S. Entry	1	3
...oks for D. Entry	1	3
...	3	0
...	2	6

CHAMBERS'S EDUCATIONAL COURSE—EDITED BY W. AND R. CHAMBERS.

EUCLID'S ELEMENTS

OF

PLANE GEOMETRY

EXPLICITLY ENUNCIATED

BY

J. PRYDE, F.E.I.S.

WILLIAM AND ROBERT CHAMBERS
LONDON AND EDINBURGH
1860

Edinburgh:
Printed by W. and R. Chambers.

PREFACE.

In issuing this New Edition of Euclid, the Publishers believe that they have succeeded in introducing several very important improvements never before attained. These improvements may be shortly summed up as follows: The particular Enunciation of each Proposition has been changed and simplified, so as to enable the Pupil to distinguish at a glance between what is *given* and what has *to be proved* in each—thus making the Propositions much more easily understood and sooner mastered. As a further aid to the Pupil, a blank space has been left between each *step* or *clause* of the *reasoning* in every Proposition throughout the book.

Particular care has been taken to render the language clear both in the Enunciations and Demonstrations; and in addition to Euclid's Demonstrations, which have been retained for the sake of those who prefer them, several of his Propositions have been also demonstrated in a different way, which, it is believed, will be found to be both more clear and elegant.

In addition to the First Six Books of Euclid's Elements, there are given a Book on the Quadrature of the Circle, a Book on Maxima and Minima, and a Treatise on Plane Trigonometry

The Fifth Book, as amended by the late Professor Playfair, is given, with the addition of a Supplementary Fifth Book, of a simpler but not less logical nature.

To render the whole more complete, several important Corollaries have been added; and, lastly, a number of instructive Exercises are annexed to each Book, to enable the Teacher to test the skill and inventive powers of the Pupil.

Edinburgh, 1860.

CONTENTS.

INTRODUCTION.

THE science of Geometry is the development of a system of truths in reference to lines, angles, surfaces, and solids. Plane Geometry treats of lines, surfaces, and angles situated in a plane. The truths of Geometry are deduced one from another, in a regular concatenation of propositions, the foundation or starting-point being certain first principles called axioms, or self-evident truths, and definitions derived from the most obvious properties of figures. This being the case, it is only necessary that the student appeal to the dictates of his own mind, in order to see whether the various propositions be true; he is therefore not under the necessity of taking anything on the testimony of another, or even of appealing to experiment, as in the case of the physical sciences, but can see clearly for himself at every step whether what is affirmed be true or not. The only arbitrary thing in Geometry is the names given to the various figures, and these are derived by the ordinary methods of forming words. Thus, a figure having three sides is called a triangle or a trilateral figure, from its having three angles or three sides; a figure having four sides is called a quadrilateral figure; and, in like manner, a figure having many angles or sides, is called a polygon or a multilateral figure.

Since every proposition in Geometry is used as an axiom ever after it has been demonstrated, it is evident that the progress of the student can only be secured by his learning perfectly every single proposition as he proceeds; and if he has patience to do so for some time, he will acquire a habit which will be of very great advantage to him in after-life, not only in the study of Mathematics, but in everything else; for he will never afterwards be satisfied with any assertion, whether made by himself or others, without knowing distinctly the grounds on which it rests. Thus,

in learning Geometry in a proper manner, he will at the same time be laying a sure foundation for excellence in any branch of study that he may afterwards undertake. Besides, the facts thus acquired are the key, as it were, to all the exact sciences; for science is only rendered exact by bringing it under the domain of Mathematics. Thus, without a knowledge of Geometry, it is impossible to learn Mechanical Philosophy, Surveying, Navigation, or Astronomy.

The propositions of Geometry are each composed of four distinct parts—namely, the General Enunciation, the Particular Enunciation, the Construction, and the Demonstration. In this edition, these parts have been so separated as to make it evident at a glance where the one ends and the other begins, so that the pupil may always know exactly with what part he is engaged, and be enabled to make himself master of the one before he proceeds to the other. The General Enunciation is that statement which stands in plain language at the top; the Particular Enunciation is the same repeated with reference to the diagram; next comes the Construction, except when the proposition is of a kind that requires none; and lastly, Demonstration. As the force of the reasoning can only be appreciated by having a clear conception of the previous parts, the pupil should master each part in its order.

ELEMENTS OF PLANE GEOMETRY.

FIRST BOOK.

Mathematics is that branch of science which treats of Measurable Quantity.

Geometry is a branch of mathematics which treats of that species of quantity called Magnitude.

Magnitudes are of one, two, or three dimensions; as lines, surfaces, and solids. They have no material existence, but they may be represented by diagrams.

That branch of Geometry which refers to magnitudes described upon a plane, is called Plane Geometry.

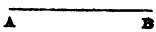
DEFINITIONS OF MAGNITUDES.

1. A *point* is that which has position, but not magnitude.

2. A *line* is length without breadth.

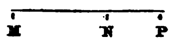
Hence the extremities of a line are points; and the intersections of lines are also points.

Lines are named by two letters placed one at each extremity. Thus, the line drawn here is named the line AB.



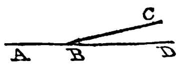
3. Lines which cannot coincide in two points, without coinciding altogether, are called *straight* lines, as AB.

A part of a line is called a *segment* of it; thus MN, a part of the line MP, is called a segment of MP, and NP is another segment of it.



Hence two straight lines cannot enclose a space. Neither can two straight lines have a common segment; that is, they cannot coincide in part, without coinciding altogether.

Thus if the lines ABC and ABD have a common segment, AB, they cannot both be straight lines.

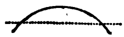


4. A *crooked* or *broken* line is composed of two or more straight lines.

5. A line of which no part is a straight line, is called a *curved line*, a *curve line*, or *curve*.

6. A *mixed* line is one composed of straight and curve lines.

7. A *convex* or *concave* line is such that it cannot be cut by a straight line in more than two points; the *concavity* of the intercepted portion is turned towards the straight line, and the *convexity* from it.

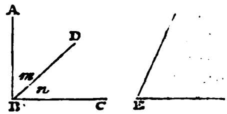


8. A *superficies* or *surface* has only length and breadth.

The extremities of a superficies are lines; and the intersections of one superficies with another are also lines.

9. A *plane superficies* is that in which any two points being taken, the straight line between them lies wholly in that superficies.

10. A *plane rectilineal angle* is the inclination of two straight lines which meet together, but are not in the same straight line.



The lines containing an angle are called its *sides*, and the point at which it is formed the *angular point*.

Thus the lines BC and BD are the sides of the angle contained by them, and B is the angular point.

An angle is named by the three letters used for naming its sides, the letter at the angular point being placed in the middle.

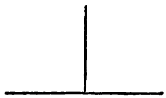
Thus the angle CBD or DBC is the angle contained by the lines BC and BD; and ABC or CBA, the angle contained by AB and BC.

When only one angle is formed at a point, it may be named by only one letter, as the angle at E.

An angle may also be named by means of a small letter placed within the angle, and near to the angular point.

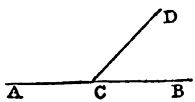
Thus, angle DBC may be called angle *n*; and ABD, angle *m*.

11. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a *right angle;* and the straight line which stands on the other is called a *perpendicular* to it.



12. The two adjacent angles which one straight line makes with another on the same side of it are called *supplementary* angles.

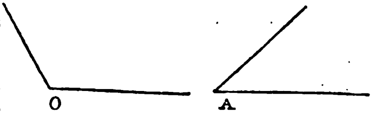
Thus the angles ACD and BCD, which CD makes with AB, are said to be supplementary.



The difference between an angle and a right angle is called its *complement;* thus, if ABC (fig. to def. 10) is a right angle, *m* and *n* are *complementary* angles.

13. An *obtuse* angle is that which is greater than a right angle, as O.

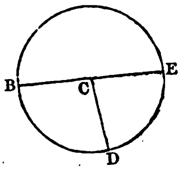
14. An *acute* angle is that which is less than a right angle, as A.



15. When an angle is not a right angle, it is said to be *oblique;* also two angles are said to be of the same *species* when they are both less or both not less than a right angle.

16. A *figure* is that which is enclosed by one or more boundaries. The space contained within the boundary of a plane figure is called its *surface;* and its surface in reference to that of another figure, with which it is compared, is called its *area.*

17. A *circle* is a plane figure contained by one line, which is called the *circumference,* and is such, that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.



18. This point is called the *centre* of the circle, as C.

Any other point within the circle is an *eccentric* point.

19. A line drawn from the centre to the circumference of a circle is called a *radius,* as CD, CB, or CE.

20. A *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference, as BE.

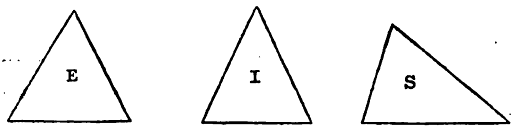
21. A *semicircle* is the figure contained by a diameter, and the part of the circumference cut off by the diameter.

22. *Rectilineal* figures are those which are contained by straight lines.

23. *Trilateral* figures, or triangles, are contained by three straight lines.

24. *Quadrilateral* figures are contained by four straight lines.

25. *Multilateral* figures, or *polygons,* are contained by more than four straight lines.



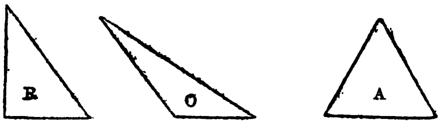
26. Of three-sided figures, an *equilateral* triangle is that which has three equal sides, as E.

27. An *isosceles* triangle is that which has only two sides equal, as I.

28. A *scalene* triangle is that which has three unequal sides, as S.

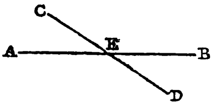
29. A *right-angled* triangle is that which has a right angle, as R.

30. An *obtuse-angled* triangle is that which has an obtuse angle, as O.



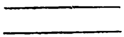
31. An *acute-angled* triangle is that which has three acute angles, as A.

32. When two straight lines intersect each other, the opposite angles at this point are called *vertical* angles; thus AEC and DEB are vertical angles; and also AED and CEB.



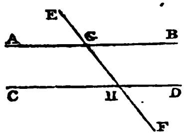
33. When one side of a triangle is produced, the outward angle thus formed is called the *exterior* angle; thus ACD (see fig. Prop. xvii. Book I.) is an exterior angle of the triangle ABC.

34. *Parallel* straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.



35. A straight line cutting or falling upon two parallel lines is called a *secant*, as EF.

36. The angles contained by a secant and two parallels lying between the parallels are called *interior* angles, as AGH, BGH, CHG, and GHD; and the angles contained by a secant and parallel lines lying towards the outside of these lines are called *exterior* angles, as AGE, BGE, CHF, and DHF.



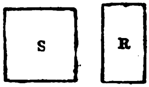
37. *Alternate* angles are those interior angles that lie on opposite sides of the secant, as AGH and DHG, or BGH and CHG.

38. When any rectilineal figure has all its angles equal, it is said to be *equiangular;* and when all its sides are equal, it is said to be *equilateral.*

39. When the angles of one rectilineal figure are respectively equal to those of another, the figures are said to be *equiangular;*

and if their sides are respectively equal, they are said to be *equilateral*.

40. A *square* is a four-sided figure which has all its sides equal, and all its angles right angles, as S.



41. An *oblong* or *rectangle* is a four-sided figure which has all its angles right angles, but has not all its sides equal, as R.

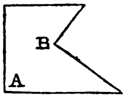
42. A *rhombus* is a quadrilateral which has all its sides equal, but its angles are not right angles, as B.

43. A *parallelogram* is a quadrilateral, of which the opposite sides are parallel, as P.

44. A *trapezoid* is a quadrilateral, having only two sides parallel, as D.

45. All other four-sided figures besides these are called *trapeziums*.

46. When an angle of a rectilineal figure is less than two right angles, it is called a *salient* angle, as A; and when greater than two right angles, it is said to be *re-entrant*, as B.



47. Any side of a rectilineal figure may be called the *base*. In a right-angled triangle, the side opposite to the right angle is called the *hypotenuse;* either of the other two sides, the *base;* and the other, the *perpendicular*. In an isosceles triangle, the side which is neither of the equal sides is called the *base*. The angular point opposite to the base of a triangle is called the *vertex;* and the angle at the vertex, the *vertical* angle.

48. The *altitude* of a triangle, or a parallelogram, is the length of a perpendicular drawn from the opposite angle or side upon the base.

49. A straight line joining two of the opposite angular points of a quadrilateral is called a *diagonal*.

POSTULATES.

1. Let it be granted that a straight line may be drawn from any one point to any other point;

2. That a terminated straight line may be produced to any length in a straight line;

3. And that a circle may be described from any centre, and with any radius.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are double of the same, are equal.

7. Things which are halves of the same, are equal.

8. Magnitudes which coincide with one another—that is, which exactly fill the same space—are equal to one another.

9. The whole is greater than its part.

10. All right angles are equal to one another.

11. Two straight lines cannot be drawn through the same point, parallel to the same straight line, without coinciding with one another.

12. It is possible for another figure to exist, equal in every respect to any given figure.

13. If two things be equal, and a third be greater than one of them, it is also greater than the other.

14. If two things be equal, and one of them be greater than a third, the other is also greater than the third.

15. If there be three magnitudes, of which the first is greater than the second, and the second is greater than the third, still more is the first greater than the third.

DEFINITIONS OF TERMS.

1. A *proposition* is a portion of science, and is a theorem, a problem, or a lemma.

2. A *theorem* is a truth which is established by a demonstration.

3. A *problem* either proposes something to be effected, as the construction of a figure; or it is a question that requires a solution.

4. A *lemma* is a subordinate truth previously established, to be employed in the demonstration of a theorem, or the solution of a problem.

5. A *hypothesis* is an assumption made without proof, either in the enunciation of a proposition, or in the course of a demonstration.

6. A *corollary* is a consequence easily deduced from one or more propositions.

7. A *scholium* is a remark on one or more propositions, which explains their application, connection, limitation, extension, or some other important circumstance in their nature.

8. A *demonstration* is a process of reasoning, and is either direct or indirect.

9. A *direct* demonstration is a regular process of reasoning from the premises to the conclusion.

10. An *indirect* demonstration establishes a proposition, by

proving that any hypothesis contrary to it is contradictory or absurd; and it is, therefore, sometimes called a *reductio ad absurdum*.

11. The *enunciation* is the statement or expression of the proposition; the *particular* enunciation is its statement in reference to a particular figure or figures.

12. The *construction* is an operation by which lines are drawn or points determined according to certain conditions.

13. The *data* or *premises* of a proposition are the relations or conditions granted or given, from which new relations are to be deduced, or a construction to be effected.

EXPLANATION OF SIGNS, ABBREVIATIONS, &c.

The signs + and − are respectively called *plus* and *minus*. The former indicates addition; thus, A + B is the sum of A and B, and A + B + C is the sum of A, B, and C: the latter indicates subtraction; thus, A − B is the excess of A above B; so A + B − C is the excess of A + B above C.

The signs $>$ and $\angle$ are respectively called *greater* and *less*. Thus, $A > B$, means that A is greater than B; and $A \angle B$, that A is less than B.

The sign =, called *equal*, is the sign of equality; thus, A = B implies that A is equal to B.

The small letters *a*, *b*, *c*, *m*, *n*, *p*, *q*, &c., are commonly used to denote numbers. A number placed before any quantity serves as a multiplier to it: thus, 3A means three times A; *m*A means *m* times A, or A taken as often as there are units in *m*.

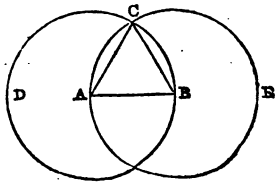
The square described on a line A is sometimes expressed by A^2, called A square; or if AB be the line, by AB^2.

The terms Proposition, Problem, Corollary, Construction, Demonstration, Preparation, &c., are respectively abbreviated, Prop., Prob., Cor., Const., Dem., Prep., &c.

PROPOSITION I. PROBLEM.

To describe an equilateral triangle upon a given finite straight line.

Given, the straight line AB; *it is required* to describe an equilateral triangle upon it.



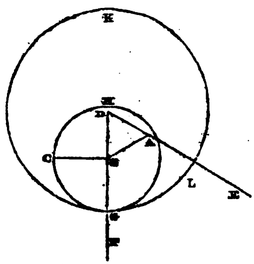
(*Const.*) From the centre A, at the distance AB, describe (Postulate 3) the circle BCD, and from the centre B, at the distance BA, describe the circle ACE; and from the point C, in which the circles cut one another, draw the straight lines (Post. 1) CA and CB, to the

points A and B; ABC shall be the equilateral triangle required.

(*Dem.*) Because the point A is the centre of the circle BCD, AC is equal (Definition 17) to AB; and because the point B is the centre of the circle ACE, BC is equal to BA; but it has been proved that CA is equal to AB; therefore CA and CB are each of them equal to AB; therefore (Axiom 1) CA is equal to CB; wherefore CA, AB, and BC, are equal to one another; and the triangle ABC (Def. 26) is therefore equilateral, and it is described upon the given straight line AB.

PROPOSITION II. PROBLEM.

From a given point, to draw a straight line equal to a given straight line.



Given, the point A, and the straight line BC; *it is required* to draw from the point A a straight line equal to BC.

(*Const.*) From the point A to B draw (Post. 1) the straight line AB; and upon it describe (I. 1) the equilateral triangle DAB, and produce (Post. 2) the straight lines DA and DB, to E and F; from the centre B, with the radius BC, describe (Post. 3) the circle CGH, and from the centre D, at the distance DG, describe the circle GKL. AL shall be equal to BC.

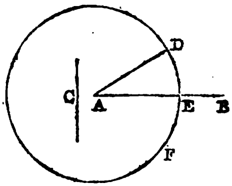
(*Dem.*) Because the point B is the centre of the circle CGH, BC is equal (Def. 17) to BG; and because D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal, being sides of an equilateral triangle; therefore the remainder AL is equal to the remainder BG (Ax. 3). But it has been shewn that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the same are equal to one another; therefore the straight line AL is equal to BC. Wherefore from the given point A a straight line AL has been drawn equal to the given straight line BC.

PROPOSITION III. PROBLEM.

From the greater of two given straight lines to cut off a part *equal to the less.*

Given, AB and C two straight lines, whereof AB is the greater. *It is required* to cut off from AB, the greater, a part equal to C, the less.

(*Const.*) From the point A draw (I. 2) the straight line AD equal to C; and from the centre A, and with the radius AD, describe (Post. 3) the circle DEF.

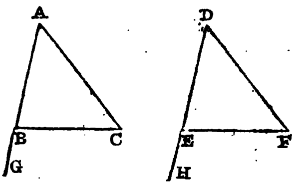


(*Dem.*) Because A is the centre of the circle DEF, AE is equal to AD; but the straight line C is likewise equal to AD; whence AE and C are each of them equal to AD; wherefore the straight line AE is equal to C (Ax. 1), and from AB, the greater of two straight lines, a part AE has been cut off equal to C, the less.

PROPOSITION IV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, their bases, or third sides, shall be equal; and the two triangles shall be equal; and their other angles shall be equal, each to each, namely, those to which the equal sides are opposite. Or, if two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

Given, ABC and DEF, two triangles which have the two sides AB and AC, equal to the two sides DE and DF, each to each, namely, AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, *to prove that* the base BC shall be equal to the base EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal sides are opposite, shall be equal, each to each; namely, the angle ABC to the angle DEF, and the angle ACB to DFE.



(*Const.*) For, if the triangle ABC be applied to the triangle DEF, so that the point A may be on D, and the straight line AB upon DE, (*Dem.*) the point B shall coincide with the point E, because AB is equal to DE; and AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF; wherefore also the point C shall coincide with the point F, because AC is equal to DF: but the point B coincides with the point E; wherefore the base BC shall coincide with the base EF (Def. 3), and shall be

equal to it. Therefore, also, the whole triangle ABC shall coincide with the whole triangle DEF, and be equal to it; and the remaining angles of the one shall coincide with the remaining angles of the other, and be equal to them; namely, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Cor.—If the equal sides AB and DE be produced to G and H, the angles GBC and HEF below the base will also be equal. For BG and EH will coincide, and therefore the angle GBC is equal to the angle HEF.

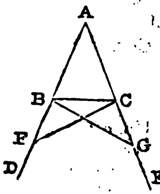
PROPOSITION V. THEOREM.

The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Given, ABC an isosceles triangle, of which the side AB is equal to AC, and let the straight lines AB and AC be produced to D and E, *to prove that* the angle ABC shall be equal to the angle ACB, and the angle CBD to the angle BCE.

(*Const.*) In BD take any point F, and from AE, the greater, cut off AG equal (I. 3) to AF, the less, and join FC and GB.

(*Dem.*) Because AF is equal to AG, and AB to AC, the two sides FA and AC are equal to the two GA and AB, each to each; and they contain the angle FAG common to the two triangles AFC and AGB; therefore the base FC is equal (I. 4) to the base GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite; namely, the angle ACF to the angle ABG, and the angle AFC to the angle AGB. And because the whole AF is equal to the whole AG, and the part AB to the part AC, the remainder BF shall be equal (Ax. 3) to the remainder CG; and FC was proved to be equal to GB; therefore the two sides BF and FC are equal to the two CG and GB, each to each; but the angle BFC is equal to the angle CGB; wherefore the triangles BFC and CGB are equal, and their remaining angles are equal, to which the equal sides are opposite; therefore the angle FBC is equal to the angle GCB, and (I. 4, cor.) the angle ABC is equal to the angle ACB.



Cor.—Hence every equilateral triangle is also equiangular.

Scholium.—This proposition may be very simply demonstrated by bisecting the vertical angle A by a line cutting the base. Then (I. 4) there are two triangles equal in every respect, and *therefore* the angles at the base are equal. And by the corollary

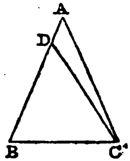
to Proposition fourth, the angles on the other side of the base are equal. It is evident that some line will bisect the vertical angle; and although the method of doing it is not known till Problem 9 be solved, yet this does not affect the truth of the demonstration.

PROPOSITION VI. THEOREM.

If two angles of a triangle be equal to one another, the sides which subtend, or are opposite to, those angles, shall also be equal to one another.

Given, ABC a triangle having the angle ABC equal to the angle ACB, *to prove that* the side AB is also equal to the side AC.

(*Const.*) For, if AB be not equal to AC, one of them is greater than the other. Let AB be the greater, and from it cut (I. 3) off DB equal to AC, the less, and join DC.



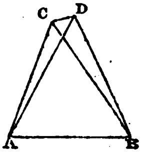
(*Dem.*) Because in the triangles DBC and ACB, DB is equal to AC, and BC common to both, the two sides DB and BC are equal to the two AC and CB, each to each; but the angle DBC is also equal to the angle ACB; therefore the base DC is equal to the base AB, and the triangle DBC is equal to the triangle ACB (I. 4), the less equal to the greater; which is absurd. Therefore AB is not greater than AC, and in the same manner it may be shewn that AB is not less than AC; that is, AB is equal to AC.

COR.—Hence every equiangular triangle is also equilateral.

PROPOSITION VII. THEOREM.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated in the other extremity, equal to one another.

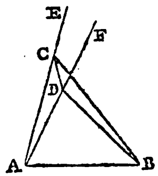
Given, two triangles ACB and ADB, upon the same base AB, and upon the same side of it, which have their sides CA and DA, terminated in A equal to one another, *to prove that* their sides CB and DB, terminated in B, cannot be equal to one another.



Join CD; then, in the case in which the vertex of each of the triangles is without the other triangle, because AC is equal to AD, the angle ACD is equal (I. 5) to the angle ADC. But the angle ACD is greater than the angle BCD; therefore the angle ADC is

greater also than BCD; still greater, then, is the angle BDC than the angle BCD. Again, if CB were equal to DB, the angle BCD would be equal to the angle BDC (I. 5), but the angle BDC has been shewn to be greater than the angle BCD; therefore BC is not equal to BD.

But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD, to E, F; then, because AC is equal to AD in the triangle ACD, the angles ECD and FDC, upon the other side of the base CD, are equal to one another, but the angle ECD is greater than the angle BCD; wherefore the angle FDC is likewise greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, if CB were equal to DB, the angle BDC would be equal to the angle BCD; but the angle BDC has been proved to be greater than the angle BCD; therefore the side BC is not equal to BD.



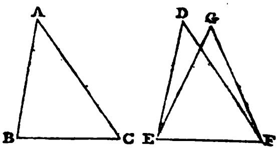
The case in which the vertex of one triangle is upon a side of the other, needs no demonstration.

PROPOSITION VIII. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides of the other. Or, if the three sides of one triangle be respectively equal to those of another, the triangles are equal in every respect, and have the angles equal that are opposite the equal sides.

Given, ABC and DEF, two triangles having the two sides AB and AC, equal to the two sides DE and DF, each to each—namely, AB to DE, and AC to DF; and also the base BC equal to the base EF; *to prove that* the angle BAC is equal to the angle EDF.

(*Const.*) For, if the triangle ABC be applied to the triangle DEF, so that the point B be on E, and the straight line BC upon EF, (*Dem.*) the point C shall also coincide with the point F, because BC is equal to EF; and BC coinciding with EF, therefore BA and AC shall coincide with ED and DF; for, if BA and CA do not coincide with ED and FD, but have a different situation as EG and FG, then, upon the same base EF, and upon the same side of it, there can be two triangles EDF and EGF, that have their sides which are terminated in one



extremity of the base equal to one another, and likewise their sides terminated in the other extremity. But this is impossible (I. 7); therefore if the base BC coincides with the base EF, the sides BA and AC cannot but coincide with the sides ED and DF; wherefore, likewise, the angle BAC coincides with the angle EDF, and is equal (Ax. 8) to it.

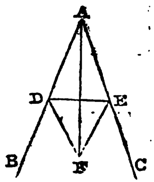
*** See Appendix—Proposition (A.)

PROPOSITION IX. PROBLEM.

To bisect a given rectilineal angle—that is, to divide it into two equal angles.

Given, the rectilineal angle BAC; *it is required* to bisect it.

(*Const.*) Take any point D in AB, and from AC cut (I. 3) off AE equal to AD; join DE, and upon it describe (I. 1) an equilateral triangle DEF; then join AF; the straight line AF bisects the angle BAC.



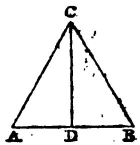
(*Dem.*) Because AD is equal to AE, and AF is common to the two triangles DAF and EAF; the two sides DA and AF are equal to the two sides EA and AF, each to each; but the base DF is also equal to the base EF; therefore the angle DAF is equal (I. 8) to the angle EAF; wherefore the given rectilineal angle BAC is bisected by the straight line AF.

PROPOSITION X. PROBLEM.

To bisect a given finite straight line—that is, to divide it into two equal parts.

Given, the straight line AB; *it is required* to divide it into two equal parts.

(*Const.*) Describe (I. 1) upon it an equilateral triangle ABC, and bisect (I. 9) the angle ACB by the straight line CD. AB is cut into two equal parts in the point D.



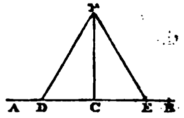
(*Dem.*) Because AC is equal to CB, and CD common to the two triangles ACD and BCD; the two sides AC and CD are equal to the two BC and CD, each to each; but the angle ACD is also equal to the angle BCD; therefore the base AD is equal to the base DB (I. 4), and the straight line AB is divided into two equal parts in the point D.

PROPOSITION XI. PROBLEM.

To draw a straight line at right angles to a given straight line, from a given point in the same.

Given, a straight line AB, and a point C in it; *it is required* to draw a straight line from the point C at right angles to AB.

(*Const.*) Take any point D in AC, and (I. 3) make CE equal to CD, and upon DE describe (I. 1) the equilateral triangle DFE, and join FC; the straight line FC, drawn from the given point C, is at right angles to the given straight line AB.

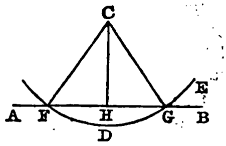


(*Dem.*) Because DC is equal to CE, and FC common to the two triangles DCF, ECF; the two sides DC and CF are equal to the two EC and CF, each to each; but the base DF is also equal to the base EF; therefore the angle DCF is equal (I. 8) to the angle ECF; and they are adjacent angles; therefore (Def. 11) each of the angles DCF and ECF is a right angle. Wherefore, from the given point C, in the given straight line AB, FC has been drawn at right angles to AB.

PROPOSITION XII. PROBLEM.

To draw a straight line perpendicular to a given straight line of an unlimited length, from a given point without it.

Given the straight line AB, which may be produced to any length both ways, and a point C without it. *It is required* to draw a straight line perpendicular to AB from the point C.



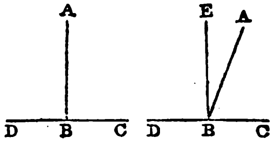
(*Const.*) Take any point D upon the other side of AB, and from the centre C, with the radius CD, describe (Post. 3) the circle EGF meeting AB in F and G; bisect (I. 10) FG in H, and join CF, CH, and CG; the straight line CH, drawn from the given point C, is perpendicular to the given straight line AB.

(*Dem.*) Because FH is equal to HG, and HC common to the two triangles FHC and GHC, the two sides, FH and HC, are equal to the two, GH and HC, each to each; now, the base CF is also equal (Def. 17) to the base CG; therefore the angle CHF is equal (I. 8) to the angle CHG; and they are adjacent angles; therefore each of them is a right angle, and CH is perpendicular to AB (Def. 11), and it is drawn from the point C, which was required to be done.

PROPOSITION XIII. THEOREM.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Given the straight line AB making with CD, upon one side of it, the angles CBA and ABD; *to prove that* these are either two right angles, or are together equal to two right angles.



(*Dem.*) For, if the angle CBA be equal to ABD, each of them is a right angle (Def. 11); but, if not, (*Const.*) *from the point B draw BE at right angles* (I. 11) *to CD;* therefore the angles CBE and EBD are two right angles; and because CBE is equal to the sum of the two angles CBA and ABE, add the angle EBD to each of these equals; therefore the two angles CBE and EBD are equal (Ax. 2) to the three angles CBA, ABE, and EBD. Again, because the angle DBA is equal to the two angles DBE and EBA, add to these equals the angle ABC; therefore the two angles DBA and ABC are equal to the three angles DBE, EBA, and ABC; but the two angles CBE and EBD have been demonstrated to be equal to the same three angles; and things that are equal to the same are equal (Ax. 1) to one another; therefore the angles CBE and EBD are equal to the angles DBA and ABC; but CBE and EBD are two right angles; therefore DBA and ABC are together equal to two right angles.

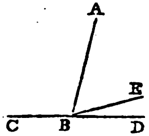
The truth of this proposition may be easily shewn as follows: The angle DBA is greater than a right angle by the angle EBA, and the angle ABC is less than a right angle by the same angle EBA; therefore their sum is two right angles.

COR.—Hence if one of a pair of supplementary angles be equal to one of another pair, the remaining angles must be equal.

PROPOSITION XIV. THEOREM.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Given that at the point B in the straight line AB, the two straight lines BC and BD, upon the opposite sides of AB, make the adjacent angles ABC and ABD equal together to two right angles. *To prove that* BD is in the same straight line with CB.



(*Const.*) For, if BD be not in the same straight line with CB, let BE be in the same straight line with it; then (*Dem.*) because the straight line AB makes angles with the straight line

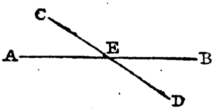
CBE, upon one side of it, therefore the angles ABC and ABE are together equal (I. 13) to two right angles; but the angles ABC and ABD are likewise together equal to two right angles; therefore the angles CBA and ABE are equal to the angles CBA and ABD: take away the common angle CBA, and the remaining angle ABE is equal (Ax. 3) to the remaining angle ABD, the less to the greater, which is impossible; therefore BE is not in the same straight line with BC. And, in like manner, it may be demonstrated that no other can be in the same straight line with it but BD, which, therefore, is in the same straight line with CB.

PROPOSITION XV. THEOREM.

If two straight lines cut one another, the vertical or opposite angles shall be equal.

Given the two straight lines AB and CD, cutting one another in the point E; *to prove that* the angle AEC shall be equal to the angle DEB, and CEB to AED.

(*Dem.*) For the angles CEA and AED, which the straight line AE makes with the straight line CD, are together equal (I. 13) to two right angles; and the angles AED and DEB, which the straight line DE makes with the straight line AB, are also together equal to two right angles; therefore the two angles CEA and AED are equal to the two AED and DEB. Take away the common angle AED, and the remaining angle CEA is equal (Ax. 3) to the remaining angle DEB. In the same manner it can be demonstrated that the angles CEB and AED are equal.



COR. 1.—From this it is manifest that, if two straight lines cut one another, the angles which they make at the point of their intersection, are together equal to four right angles.

COR. 2.—And hence, all the consecutive angles made by any number of lines meeting in one point, are together equal to four right angles.

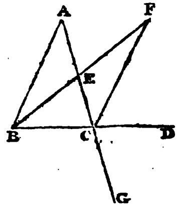
PROPOSITION XVI. THEOREM.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Given a triangle ABC, and its side BC produced to D, *to prove that* the exterior angle ACD is greater than either of the interior opposite angles CBA or BAC.

(*Const.*) Bisect (I. 10) AC in E, join BE, and produce BE to F, making EF equal to BE; join also FC, and produce AC to G.

(*Dem.*) Because AE is equal to EC, and BE to EF; AE and EB are equal to CE and EF, each to each; and the angle AEB is equal to the angle CEF (I. 15), because they are opposite vertical angles; therefore (I. 4) the base AB is equal to the base CF, and the triangle AEB to the triangle CEF, and the remaining angles to the remaining angles, each to each, to which the equal sides are opposite; wherefore the angle BAE is equal to the angle ECF; but the angle ECD is greater than the angle ECF; therefore (Ax. 14) the angle ECD, that is, ACD, is greater than BAE. In the same manner, if the side BC be bisected, and a similar construction made below the base, it may be demonstrated that the angle BCG, that is (I. 15), the angle ACD, is greater than the angle ABC; therefore the exterior angle ACD is greater than either the angle CAB or ABC.

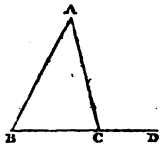


PROPOSITION XVII. THEOREM.

Any two angles of a triangle are together less than two right angles.

Given any triangle ABC; *to prove that* any two of its angles together are less than two right angles.

(*Const.*) Produce BC to D; (*Dem.*) and because ACD is the exterior angle of the triangle ABC, ACD is greater than the interior and opposite angle ABC (I. 16); to each of these add the angle ACB; therefore the angles ACD and ACB (Ax. 4) are greater than the angles ABC and ACB; but ACD and ACB are together equal to two right angles (I. 13); therefore the angles ABC and BCA are less than two right angles. In like manner it may be demonstrated, that BAC and ACB, as also CAB and ABC, are less than two right angles.



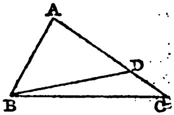
COR.—Hence there cannot be two perpendiculars drawn from a point to a line.

PROPOSITION XVIII. THEOREM.

The greater side of every triangle has the greater angle opposite to it.

Given a triangle ABC, of which the side AC is greater than the side AB; *to prove that* the angle ABC is also greater than the angle BCA.

(*Const.*) Because AC is greater than AB, make (I. 3) AD equal to AB, and join BD; (*Dem.*) and because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle DCB (I. 16); but ADB is equal to ABD (I. 5), because the side AB is equal to the side AD; therefore (Ax. 15) the angle ABD is likewise greater than the angle ACB; wherefore still more is the angle ABC greater than ACB.



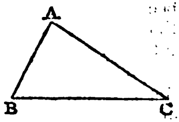
COR.—The perpendicular from a point to a straight is the least line that can be drawn from that point to it. For if the angle B were a right angle, the angle C would be less than a right angle (I. 17), therefore the perpendicular AB would be less than AC.

PROPOSITION XIX. THEOREM.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Given a triangle ABC, of which the angle ABC is greater than the angle BCA; *to prove that* the side AC is likewise greater than the side AB.

(*Dem.*) For, if AC be not greater than AB, AC must either be equal to AB, or less than it; AC is not equal to AB, because then the angle ABC would be equal to the angle ACB (I. 5); but it is not; therefore AC is not equal to AB; neither is AC less than AB; because then the angle ABC would be less than the angle ACB (I. 18); but it is not; therefore the side AC is not less than AB; and it has been shewn that it is not equal to AB; therefore AC is greater than AB.

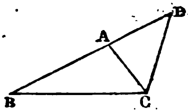


PROPOSITION XX. THEOREM.

Any two sides of a triangle are together greater than the third side.

Given a triangle ABC; *to prove that* any two sides of it together are greater than the third side; namely, the sides BA and AC greater than the side BC; AB and BC greater than AC; and BC and CA greater than AB.

(*Const.*) Produce BA to the point D, and make (I. 3) AD equal to AC; and join DC.



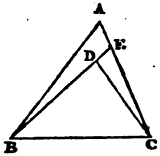
(*Dem.*) Because DA is equal to AC, the angle ADC is equal to ACD (I. 5); but the angle BCD is greater than the

angle ACD; therefore (Ax. 14) the angle BCD is greater than the angle ADC; and because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle has the greater side opposite to it (I. 19); therefore the side DB is greater than the side BC; but DB is equal to BA and AC; therefore the sides BA and AC are greater than BC. In the same manner it may be demonstrated, that the sides AB and BC are greater than CA, and BC and CA greater than AB.

PROPOSITION XXI. THEOREM.

If from the ends of one side of a triangle, there be drawn two straight lines to a point within the triangle, these two lines shall be less than the other two sides of the triangle, but shall contain a greater angle.

Given the two straight lines BD and CD, drawn from B and C, the ends of the side BC of the triangle ABC, to the point D within it; *to prove that* BD and DC are less than the other two sides BA and AC of the triangle, but contain an angle BDC greater than the angle BAC.



(*Const.*) Produce BD to E; (*Dem.*) and because two sides of a triangle are greater than the third side, the two sides BA and AE of the triangle ABE are greater than BE. To each of these add EC; therefore the sides BA and AC are greater than BE and EC. Again, because the two sides CE and ED of the triangle CED are greater than CD, add DB to each of these; therefore the sides CE and EB are greater than CD and DB; but it has been shewn that BA and AC are greater than BE and EC; still more (Ax. 16), then, are BA and AC greater than BD and DC.

Again, because the exterior angle of a triangle is greater than the interior and opposite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the same reason, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonstrated that the angle BDC is greater than the angle CEB; still more (Ax. 16), then, is the angle BDC greater than the angle BAC.

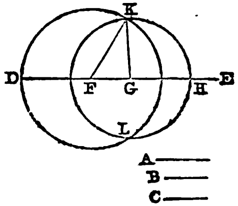
PROPOSITION XXII. PROBLEM.

To make a triangle of which the sides shall be equal to three given straight lines; but any two whatever of these lines must be greater than the third (I. 20).

Given A, B, and C, three straight lines, of which any two

whatever are greater than the third; namely, A and B greater than C, A and C greater than B, and B and C greater than A. *It is required* to make a triangle, of which the sides shall be equal to A, B, and C.

(*Const.*) Take a straight line DE terminated at the point D, but unlimited towards E, and make (I. 3) DF equal to A, FG to B, and GH equal to C; and from the centre F, with the radius FD, describe the circle DKL (Post. 3); and from the centre G, with the radius GH, describe another circle HLK, and join KF and KG; the triangle KFG has its sides equal to the three straight lines A, B, and C.

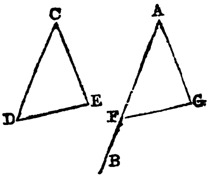


(*Dem.*) Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 17); but FD is equal to the straight line A; therefore FK (Ax. 1) is equal to A. Again, because G is the centre of the circle LKH, GH is equal to GK; but GH is equal to C; therefore also GK is equal to C, and FG is equal to B; therefore the three straight lines KF, FG, and GK, are equal to the three A, B, and C. And therefore the triangle KFG has its three sides KF, FG, and GK, equal to the three given straight lines, A, B, and C.

PROPOSITION XXIII. PROBLEM.

At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Given the straight line AB, and the point A in it, and the rectilineal angle DCE; *it is required* to make an angle at the given point A in the given straight line AB, that shall be equal to the given rectilineal angle DCE.

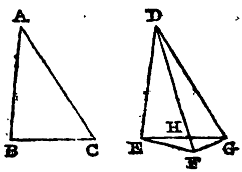


(*Const.*) Take in CD and CE, any points D and E, and join DE; and make (I. 22) the triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, and CE, so that AF be equal to CD, AG to CE, and FG to DE; (*Dem.*) and because DC and CE are equal to FA and AG, each to each, and the base DE to the base FG; the angle DCE is equal to the angle FAG (I. 8). Therefore, at the given point A in the given straight line AB, the angle FAG is made equal to the given rectilineal angle DCE.

PROPOSITION XXIV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them, of the other; the base of that which has the greater angle shall be greater than the base of the other.

Given two triangles, ABC and DEF, which have the two sides AB and AC equal to the two DE and DF, each to each— namely, AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; *to prove that* the base BC is also greater than the base EF.



(*Const.*) Of the two sides DE, DF, let DE be the side which is not greater than the other, and at the point D, in the straight line DE, make (I. 23) the angle EDG, equal to the angle BAC; and make DG equal to AC or DF (I. 3), and join EG and GF.

(*Dem.*) Because DE is not greater than DF or DG, therefore the angle DGE is not greater than DEG; but angle DHG is greater than DEG (I. 16); therefore DHG is greater than DGE, and hence the side DG or DF (I. 19) is greater than DH; therefore the line EG must lie between EF and ED, or the point F must be below EG.

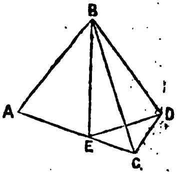
Because AB is equal to DE, and AC to DG, the two sides BA and AC are equal to the two ED and DG, each to each, and the angle BAC is equal to the angle EDG; therefore the base BC is equal to the base EG (I. 4); and because DG is equal to DF, the angle DFG is equal to the angle DGF (I. 5); but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF (Ax. 14); and still more is the angle EFG greater than the angle EGF; and because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle has the greater side opposite to it (I. 19), the side EG is therefore greater than the side EF; but EG is equal to BC; and therefore also (Ax. 15) BC is greater than EF.

Otherwise:

Given two triangles ABC and DBC, which have AB equal to DB, and BC common, but the angle ABC greater than the angle DBC; *to prove that* AC is greater than CD.

(*Const.*) Draw BE (I. 9) bisecting the angle ABD, which

will therefore fall in the greater of the two angles ABC, and cut AC in E; join ED.

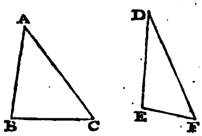


(*Dem.*) Because AB is equal to DB, and BE common to the two triangles ABE, DBE; the two sides AB and BE are equal to the two DB and BE, and the contained angle ABE is equal to the contained angle DBE, therefore (I. 4) the base AE is equal to DE; to each of these equals add EC, and AC is equal to DE and EC, but DE and EC are greater than DC (I. 20); wherefore AC is greater than DC.

PROPOSITION XXV. THEOREM.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base, shall be greater than the angle contained by the sides equal to them, of the other.

Given two triangles ABC and DEF, which have the two sides AB and AC equal to the two sides DE and DF, each to each— namely, AB equal to DE, and AC to DF; but have the base CB greater than the base EF; *to prove that* the angle BAC is likewise greater than the angle EDF.



(*Dem.*) For, if the angle BAC be not greater than EDF, the angle BAC must either be equal to EDF, or less than it; but the angle BAC is not equal to the angle EDF, because, then, the base BC would be equal to EF (I. 4), which it is not; therefore the angle BAC is not equal to the angle EDF; neither is it less; because then the base BC would be less than the base EF (I. 24), which it is not; therefore the angle BAC is not less than the angle EDF; and it was shewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF.

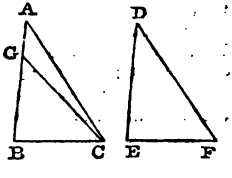
PROPOSITION XXVI. THEOREM.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—namely, either the sides adjacent to the equal angles, or the sides opposite to the equal angles in each; then shall the other sides be equal, each to each; and the triangles shall be equal; and also the third angle of the one to the third angle of the other.

Or, if two angles and a side in one triangle be respectively

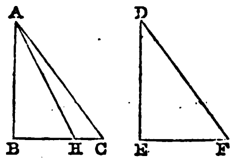
equal to two angles and a corresponding side in another triangle, the two triangles shall be equal in every respect.

Given two triangles ABC and DEF, which have the angles ABC and BCA of the one equal to the angles DEF and EFD of the other— namely, ABC to DEF, and BCA to EFD; also one side equal to one side; and first let those sides be equal which are adjacent to the angles that are equal in the two triangles— namely, BC to EF; *to prove that* the other sides shall be equal, each to each— namely, AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



(*Const.*) For, if AB be not equal to DE, one of them must be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; (*Dem.*) therefore because BG is equal to DE, and BC to EF, the two sides GB and BC are equal to the two DE and EF, each to each; and the angle GBC is equal to the angle DEF; therefore the base GC is equal to the base DF (I. 4), and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypothesis, equal to the angle BCA; wherefore also the angle BCG is equal to the angle BCA, the less to the greater, which is impossible; therefore AB is not unequal to DE; that is, it is equal to it; and BC is equal to EF; therefore the two AB and BC are equal to the two DE and EF, each to each; and the angle ABC is equal to the angle DEF; therefore the base AC is equal to the base DF, and the triangle ABC to the triangle DEF, and the third angle BAC to the third angle EDF.

Next, let the sides which are opposite to equal angles in each triangle be *given* equal to one another; namely, AB to DE; *to prove that* likewise in this case, the other sides shall be equal, AC to DF, and BC to EF; and also the third angle BAC to the third EDF.



(*Const.*) For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; (*Dem.*) and because BH is equal to EF, and AB to DE, the two sides AB and BH are equal to the two DE and EF, each to each; and they contain equal angles; therefore the base AH is equal to the base DF, and the triangle ABH to the triangle DEF, and the other angles are equal, each to each, to which the equal sides

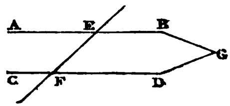
are opposite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore also the angle BHA is equal to the angle BCA; that is, the exterior angle BHA of the triangle AHC is equal to its interior and opposite angle BCA, which is impossible (I. 16); wherefore BC is not unequal to EF; that is, it is equal to it; and AB is equal to DE; therefore the two sides AB and BC are equal to the two DE and EF, each to each; and they contain equal angles; wherefore the base AC is equal to the base DF, and the triangle ABC to the triangle DEF, and the third angle BAC to the third angle EDF.

PROPOSITION XXVII. THEOREM.

If a straight line falling upon two other straight lines makes the alternate angles equal to one another, these two straight lines shall be parallel.

Given that the straight line EF, which falls upon the two straight lines AB and CD, makes the alternate angles AEF and EFD equal to one another; *to prove that* AB is parallel to CD.

(*Const.*) For, if AB and CD be not parallel, being produced, they shall meet either towards B, D, or towards A, C; let them be produced, and meet towards B, D, in the point G; (*Dem.*) therefore GEF is a triangle, and its exterior angle AEF is greater (I. 16) than the interior and opposite angle EFG; but it was given equal to it, which is impossible;



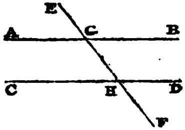
therefore AB and CD being produced, do not meet towards B, D. In like manner it may be demonstrated that they do not meet towards A, C; but those straight lines which are in the same plane, and being produced ever so far both ways do not meet, are parallel to one another (Def. 34). Therefore AB is parallel to CD.

PROPOSITION XXVIII. THEOREM.

If a straight line falling upon two other straight lines makes the exterior angle equal to the interior and opposite upon the same side of the line, or makes the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Given that the straight line EF, which falls upon the two straight lines AB and CD, makes the exterior angle EGB equal to the interior and opposite angle GHD upon the same side; or makes the interior angles on the same side BGH and GHD

together equal to two right angles; *to prove that* AB is parallel to CD.



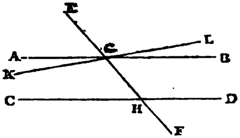
(*Dem.*) Because the angle EGB is equal to the angle GHD, and the angle EGB equal to the angle AGH (I. 15), the angle AGH is equal to the angle GHD; and they are alternate angles; therefore AB is parallel to CD (I. 27). Again, because the angles BGH and GHD are together equal (by Hyp.) to two right angles; and that AGH and BGH are also together equal to two right angles (I. 13); the angles AGH and BGH are equal to the angles BGH and GHD. Take away the common angle BGH, and there remains the angle AGH equal to the angle GHD; and they are alternate angles; therefore (I. 27) AB is parallel to CD.

PROPOSITION XXIX. THEOREM.

If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.

Given the straight line EF falling upon the parallel straight lines AB, CD; *to prove that* the alternate angles AGH, GHD, are equal to one another; and the exterior angle EGB is equal to the interior and opposite, upon the same side, GHD; and the two interior angles BGH and GHD, upon the same side, are together equal to two right angles.

(*Const.*) For if AGH be not equal to GHD, let KG be drawn making the angle KGH equal to GHD, and produce KG to L. (*Dem.*) Then KL will be parallel to CD (I. 27); but AB is also parallel to CD; therefore two straight lines are drawn through the same point G, parallel to CD, and yet not coinciding with one another, which is impossible (Ax. 11). The angles AGH and GHD are therefore not unequal; that is, they are equal to one another.



Now, the angle EGB is equal to AGH (I. 15); and AGH was proved to be equal to GHD; therefore EGB is likewise equal to GHD; add to each of these the angle BGH; therefore the angles EGB and BGH are equal to the angles BGH and GHD; but EGB and BGH are equal to two right angles (I. 13); therefore also BGH and GHD are equal to two right angles.

COR. 1.—If two lines KL and CD make, with EF, the two angles KGH and GHC, together less than two right angles, KG and CH will meet on the side of EF on which the two angles are that are less than two right angles.

(*Dem.*) For, if not, KL and CD are either parallel, or they meet on the other side of EF; but they are not parallel; for the angles KGH and GHC would then be together equal to two right angles. Neither do they meet toward the points L and D; for the angles LGH and GHD would then be two angles of a triangle, and therefore less than two right angles; but this is impossible; for the four angles KGH, HGL, CHG, and GHD are together equal to four right angles, of which the two KGH and CHG are by supposition less than two right angles; therefore the other two, HGL and GHD, are greater than two right angles. Therefore, since KL and CD are not parallel, and do not meet towards L and D, they will meet if produced towards K and C.

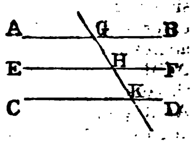
COR. 2.—If two straight lines be each perpendicular to the same line, they are parallel. For the interior angles on the same side are together equal to two right angles.

PROPOSITION XXX. THEOREM.

Straight lines which are parallel to the same straight line are parallel to one another.

Given that AB and CD are each of them parallel to EF; *to prove that* AB is also parallel to CD.

(*Const.*) Let the straight line GHK cut AB, EF, CD; and (*Dem.*) because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29). Again, because the straight line GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD; and it was shewn that the angle AGK is equal to the angle GHF; therefore also AGK is equal to GKD (Ax. 1), and they are alternate angles; therefore AB is parallel to CD (I. 27).



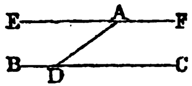
PROPOSITION XXXI. PROBLEM.

To draw a straight line through a given point parallel to a given straight line.

Given the point A and the straight line BC; *it is required* to draw a straight line through the point A parallel to the straight line BC.

(*Const.*) In BC take any point D, and join AD; and at

the point A, in the straight line AD make (I. 23) the angle DAE equal to the angle ADC; and produce the straight line EA to F.

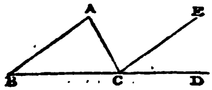


(*Dem.*) Because the straight line AD, which meets the two straight lines BC and EF, makes the alternate angles EAD, ADC equal to one another, EF is parallel to BC (I. 27). Therefore the straight line EAF is drawn through the given point A parallel to the given straight line BC.

PROPOSITION XXXII. THEOREM.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

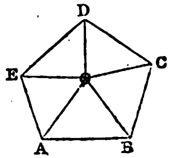
Given a triangle ABC, having one of its sides BC produced to D; *to prove that* the exterior angle ACD is equal to the two interior and opposite angles CAB and ABC, and that the three interior angles of the triangle—namely, ABC, BCA, and CAB, are together equal to two right angles.



(*Const.*) Through the point C draw CE parallel to the straight line AB (I. 31).

(*Dem.*) And because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29). Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC; but the angle ACE was shewn to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles CAB and ABC. To each of these equals add the angle ACB, and the two angles ACD and ACB are equal to the three angles CBA, BAC, and ACB; but the angles ACD and ACB are equal to two right angles (I. 13); therefore also the angles CBA, BAC, and ACB are equal to two right angles.

COR. 1.—All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.



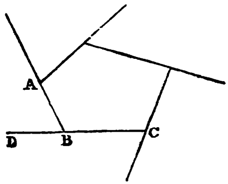
For any rectilineal figure ABCDE can be divided into as many triangles as the figure has sides, by drawing straight lines from a point O within the figure to each of its angles. And, by the preceding proposition, all the angles of these triangles are equal to twice as many right angles as there are triangles—that is, as there are sides of the figure; and the same angles are equal to the angles of the

figure, together with the angles at the point O, which is the common vertex of the triangles; that is (I. 15, Cor. 2), together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Let the sum of the interior angles be denoted by I, the number of sides by n, and a right angle by R, then

$$I + 4R = 2nR.$$

COR. 2.—All the exterior angles of any rectilineal figure are together equal to four right angles.



Because every interior angle ABC, with its adjacent exterior ABD, is equal to two right angles (I. 13); therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are sides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

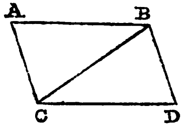
COR. 3.—If the sum of two angles in one triangle is equal to the sum of two angles in another triangle, their third angles must be equal.

COR. 4.—The two oblique angles of a right-angled triangle are together equal to a right angle.

PROPOSITION XXXIII. THEOREM.

The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.

Given AB and CD, two equal and parallel straight lines, joined towards the same parts by the straight lines, AC and BD; *to prove that* AC and BD are also equal and parallel.



(*Const.*) Join BC; (*Dem.*) and because AB is parallel to CD, and BC meets them, the alternate angles ABC and BCD are equal (I. 29); and because AB is equal to CD, and BC common to the two triangles ABC and DCB, the two sides AB and BC are equal to the two DC and CB; and the angle ABC is equal to the angle DCB; therefore the base AC is equal to the base BD (I. 4), and the triangle ABC to the triangle DCB, and

the other angles to the other angles, each to each, to which the equal sides are opposite; therefore the angle ACB is equal to the angle DBC; and because the straight line BC meets the two straight lines AC and BD, and makes the alternate angles ACB and DBC equal to one another, AC is parallel to BD (I. 27); and it was shewn to be equal to it.

COR.—If two equal perpendiculars be drawn to a straight line on the same side of it, the straight line joining their extremities is parallel to the former, and equal to the intercepted part of it.

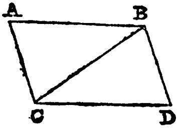
For the perpendiculars are equal and also parallel (I. 28); therefore the lines joining their extremities are equal and parallel.

PROPOSITION XXXIV. THEOREM.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it; that is, divides it in two equal parts.

Given a parallelogram ACDB, of which BC is a diagonal; *to prove that* the opposite sides and angles of the figure are equal to one another, and that the diagonal BC bisects it.

(*Dem.*) Because AB is parallel to CD, and BC meets them, the alternate angles ABC and DCB are equal to one another (I. 29); and because AC is parallel to BD, and BC meets them, the alternate angles ACB and DBC are equal to one another; wherefore the two triangles ABC and DCB have two angles ABC and BCA, in one, respectively equal to two angles DCB and CBD, in the other, each to each, and one side BC common to the two triangles, which is adjacent to their equal angles; therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other (I. 26)— namely, the side AB equal to the side CD, and AC to BD, and the angle BAC equal to the angle BDC. And because the angle ABC is equal to the angle DCB, and the angle CBD to the angle BCA, the whole angle ABD is equal to the whole angle ACD. And the angle BAC has been shewn to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another; also, the diagonal bisects it; for it was proved that the two triangles ABC and BCD are every way equal (I. 26), consequently the diagonal BC divides the parallelogram ACDB into two equal parts.



COR. 1.—Parallel lines are equidistant.

For, if from two points in one of them perpendiculars be drawn

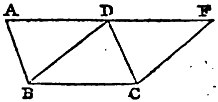
to the other, they are parallel (I. 28), and the two lines intercepted between them are parallel; therefore a parallelogram is formed, of which the perpendiculars are opposite sides, and are therefore equal.

COR. 2.—Hence two triangles or two parallelograms between the same parallels have the same altitude; and if they have the same altitude, and be on the same side of bases that are in the same straight line, they are between the same parallels. (I. 33, Cor.)

PROPOSITION XXXV. THEOREM.

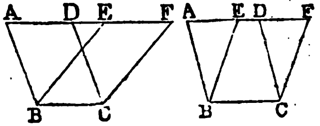
Parallelograms upon the same base, and between the same parallels, are equal to one another.

Given two parallelograms ABCD and EBCF (see the 2d and 3d figures) upon the same base BC, and between the same parallels AF, BC; *to prove that* the parallelogram ABCD is equal to the parallelogram EBCF.



(*Dem.*) If the sides AD and DF, of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D (as in fig. 1), it is plain that each of the parallelograms is double of the triangle DBC (I. 34); and they are therefore equal to one another.

But if the sides AD and EF, opposite to the base BC of the parallelograms ABCD and EBCF, be not terminated in the same point, then because ABCD is a parallelogram, AD is equal to BC; for the same reason EF is equal to BC; wherefore AD is equal to EF (Ax. 1); and DE is common; therefore (Ax. 2 or 3) the whole or the remainder AE is equal to the whole or the remainder DF; AB also is equal to DC; and the two EA and AB are therefore equal to the two FD and DC, each to each; and the exterior angle FDC is equal (I. 29) to the interior EAB; therefore the base EB is equal to the base FC, and the triangle EAB equal to the triangle FDC (I. 4); take the triangle FDC from the trapezoid ABCF, and from the same trapezoid take the triangle EAB, the remainders therefore are equal; that is (Ax. 3), the parallelogram ABCD is equal to the parallelogram EBCF.

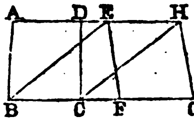


PROPOSITION XXXVI. THEOREM.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Given two parallelograms ABCD and EFGH upon equal bases BC and FG, and between the same parallels AH and BG; *to prove that* the parallelogram ABCD is equal to EFGH.

(*Const.*) Join BE and CH; (*Dem.*) and because BC is equal to FG, and FG to EH (I. 34), therefore BC is equal to EH; and they are parallels, and joined towards the same parts by the straight lines BE and CH; therefore (I. 33) EB and CH are both equal and parallel, wherefore EBCH is a parallelogram; and it is equal to ABCD (I. 35), because it is upon the same base BC, and between the same parallels BC and AH. For a like reason, the parallelogram EFGH is equal to the same parallelogram EBCH; therefore (Ax. 1) the parallelogram ABCD is equal to EFGH.

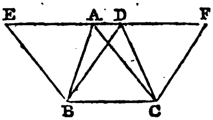


PROPOSITION XXXVII. THEOREM.

Triangles upon the same base, and between the same parallels, are equal to one another.

Given the triangles ABC and DBC upon the same base BC, and between the same parallels AD and BC; *to prove that* the triangle ABC is equal to the triangle DBC.

(*Const.*) Join AD, and produce it both ways to the points E and F, and through B draw BE parallel to CA (I. 31); and through C draw CF parallel to BD. (*Dem.*) Therefore, each of the figures EBCA and DBCF is a parallelogram; and EBCA is equal to DBCF (I. 35), because they are upon the same base BC, and between the same parallels BC and EF; and the triangle ABC is the half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34); and the triangle DBC is the half of the parallelogram DBCF, because the diagonal DC bisects it. But the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DBC.



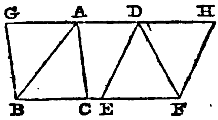
PROPOSITION XXXVIII. THEOREM.

Triangles upon equal bases, and between the same parallels, are equal to one another.

Given the triangles ABC and DEF upon equal bases BC and EF, and between the same parallels BF, AD; *to prove that* the triangle ABC is equal to the triangle DEF.

(*Const.*) Produce AD both ways to the points G and H, and through B draw BG parallel (I. 31) to CA, and through F draw FH parallel to ED. (*Dem.*) Then each of the figures

GBCA and DEFH is a parallelogram; and they are equal to one another (I. 36), because they are upon equal bases BC, EF, and between the same parallels BF, GH; and the triangle ABC is the half of the parallelogram GBCA (I. 34), because the diagonal AB bisects it; and the triangle DEF is the half of the parallelogram DEFH, because the diagonal DF bisects it. But the halves of equal things are equal (Ax. 7); therefore the triangle ABC is equal to the triangle DEF.

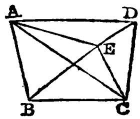


PROPOSITION XXXIX. THEOREM.

Equal triangles upon the same base, and upon the same side of it, are between the same parallels.

Given the equal triangles ABC and DBC, upon the same base BC, and upon the same side of it; *to prove that* they are between the same parallels.

(*Const.*) Join AD; AD is parallel to BC; for, if it is not, through the point A draw (I. 31) AE parallel to BC, and join EC. (*Dem.*) The triangle ABC is equal to the triangle EBC (I. 37), because it is upon the same base BC, and between the same parallels BC, AE. But the triangle ABC is given equal to the triangle BDC; therefore also, the triangle BDC is equal to the triangle EBC, the greater to the less, which is impossible; therefore AE is not parallel to BC. In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BC except AD, which therefore is parallel to it.



Cor.—This proposition is true of parallelograms.

For, if ABC, BDC be the halves of the parallelograms, then, since AD is parallel to BC, the sides of the two parallelograms passing through A and D, and being both parallel to BC, must lie in the same line with AD (Ax. 11).

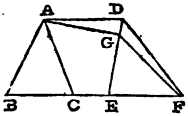
PROPOSITION XL. THEOREM.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

Given the equal triangles ABC and DEF upon the same side of equal bases BC and EF, in the same straight line BF; *to prove that* they are between the same parallels.

(*Const.*) Join AD; AD is parallel to BC; for if it is not, through A draw (I. 31) AG parallel to BF, and join

GF. (*Dem.*) The triangle ABC is equal to the triangle GEF (I. 38), because they are upon equal bases BC and EF, and between the same parallels BF, AG; but the triangle ABC is equal to the triangle DEF; therefore also the triangle DEF is equal to the triangle GEF, the greater to the less, which is impossible; therefore AG is not parallel to BF. In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BC except AD, which therefore is parallel to it.



COR.—This proposition is also true of parallelograms.

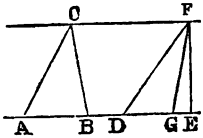
It may be proved in the same manner as the Cor. to last proposition.

PROPOSITION A. THEOREM.

Equal triangles, between the same parallels, are upon equal bases.

Given the areas of the triangles ABC and DEF, equal to each other, and that they lie between the same parallels AG, CF; *to prove that* their bases AB and DE are equal.

(*Const.*) If AB be not equal to DE, let DE be the greater, and make DG equal AB, and join G, F. (*Dem.*) Then, since AB is equal to DG, the triangles ABC and DGF are equal (I. 38); but the triangles ABC and DEF are given equal; therefore the triangle DEF is equal to DGF, the greater to the less, which is impossible. Therefore the base DE is not greater than AB, and in a similar manner it may be proved that it is not less; therefore AB is equal to DE.



COR.—This proposition is true of parallelograms.

For, if ABC and DEF be the halves of the parallelograms, they are equal, and therefore AB is equal to DE.

Scholium 1.—The preceding seven propositions, with the corollaries to the last three, are also true, when the term *altitude* is substituted for the expression *between the same parallels;* observing, that in the last three propositions it is not necessary that the two triangles or parallelograms be upon the same sides of their bases, or that their bases be in the same straight line; for when the two figures do not fulfil these conditions, other two equal to them in every respect may be constructed so as to fulfil them (I. 22), and the demonstrations will apply to the latter (I. 34, Cor. 2).

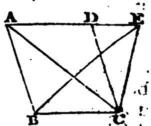
Schol. 2.—It appears from these propositions also, that of these three conditions—the equality of the bases, of the altitudes, and

of the areas of two parallelograms or triangles—if any two be given, the third will also be fulfilled; that is, if the bases and altitudes be equal, the areas are equal; if the bases and areas be equal, the altitudes are equal; and if the areas and altitudes be equal, the bases are equal.

PROPOSITION XLI. THEOREM.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Given the parallelogram ABCD and the triangle EBC upon the same base BC, and between the same parallels BC, AE; *to prove that* the parallelogram ABCD is double of the triangle EBC.



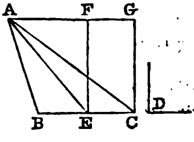
(*Const.*) Join AC. (*Dem.*) Then the triangle ABC is equal to the triangle EBC (I. 37), because they are upon the same base BC, and between the same parallels BC, AE. But the parallelogram ABCD is double of the triangle ABC (I. 34), because the diagonal AC divides it into two equal parts; wherefore ABCD is also double of the triangle EBC.

PROPOSITION XLII. PROBLEM.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Given the triangle ABC and the rectilineal angle D. *It is required* to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.

(*Const.*) Bisect (I. 10) BC in E, and join AE, and at the point E in the straight line EC make (I. 23) the angle CEF equal to D; and through A draw AG parallel to EC (I. 31), and through C draw CG parallel to EF; then FECG is a parallelogram. (*Dem.*) And because BE is equal to EC, the triangle ABE is likewise equal to the triangle AEC (I. 38), since they are upon equal bases BE and EC, and between the same parallels BC, AG; therefore the triangle ABC is double of the triangle AEC.



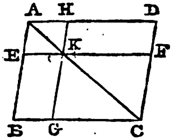
And the parallelogram FECG is likewise double of the triangle AEC (I. 41), because it is upon the same base and between the same parallels; therefore the parallelogram FECG is equal to the triangle ABC (Ax. 6), and it has one of its angles CEF equal to the given

angle D; wherefore there has been described a parallelogram FECG equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D.

PROPOSITION XLIII. THEOREM.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Given a parallelogram ABCD, of which the diagonal is AC; and let EH and FG be the parallelograms about AC, that is, through which AC passes; and BK and KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements; *to prove that* the complement BK is equal to the complement KD.

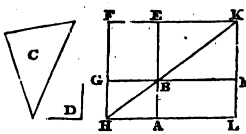


(*Dem.*) Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). And because AEKH is a parallelogram, the diagonal of which is AK, the triangle AEK is equal to the triangle AHK. For the same reason, the triangle KGC is equal to the triangle KFC. Then, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC, is equal to the triangle AHK, together with the triangle KFC. But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD (Ax. 3).

PROPOSITION XLIV. PROBLEM.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Given the straight line AB, the triangle C, and the rectilineal angle D. *It is required* to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.



(*Const.*) Make (I. 42) the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, so that BE be in the same straight line with AB, and produce FG to H; and through A draw (I. 31) AH parallel to BG or EF, and join HB. Then because the straight line HF falls upon the parallels AH, EF, the angles AHF and HFE, are together equal to two right

angles (I. 29); wherefore the angles BHF and HFE, are less than two right angles. But straight lines, which, with another straight line, make the interior angles upon the same side less than two right angles, do meet if produced far enough (I. 29, Cor.); therefore HB and FE shall meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA and GB to the points L and M. (*Dem.*) Then HLKF is a parallelogram, of which the diagonal is HK; and AG, ME, are the parallelograms about HK; and LB, BF, are the complements; therefore LB is equal to BF (I. 43). But BF is equal to the triangle C; wherefore (Ax. 1) LB is equal to the triangle C; and because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D; the angle ABM is equal to the angle D; therefore the parallelogram LB is applied to the straight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D.

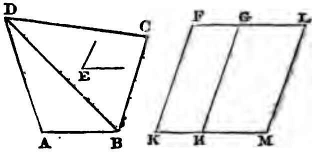
PROPOSITION XLV. PROBLEM.

To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Given the rectilineal figure ABCD, and the rectilineal angle E.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

(*Const.*) Join DB, and describe (I. 42) the parallelogram FH equal to the triangle ABD, and having the angle HKF equal to the angle E; and (I. 44) to the straight line GH apply the parallelogram GM equal to the triangle DBC, having the angle GHM equal to the angle E. (*Dem.*) And because the angle E is equal to each of the angles FKH and GHM, the angle FKH is equal to GHM;



add to each of these the angle KHG; therefore the angles FKH and KHG are equal to the angles KHG and GHM; but FKH and KHG are equal to two right angles (I. 29); therefore also KHG and GHM are equal to two right angles; therefore KH is in the same straight line (I. 14) with HM. And because the straight line HG meets the parallels KM and FG, the alternate angles MHG and HGF are equal; add to each of these the angle HGL; therefore the angles MHG and HGL are equal to the angles HGF and HGL; but the angles MHG and HGL are together equal to two right angles; wherefore also the angles HGF and HGL are equal to two right angles, and FG is therefore in the same straight line (I. 14) with GL. And

because KF is parallel to HG, and HG parallel to ML, KF is parallel to ML (I. 30); but KM and FL are parallels; wherefore KFLM is a parallelogram. And because the triangle ABD is equal to the parallelogram FH, and the triangle DBC to the parallelogram GM, the whole rectilineal figure ABCD is equal to the whole parallelogram FKML; therefore the parallelogram FKML has been described equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E.

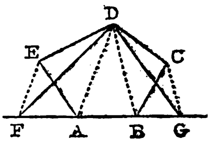
COR.—From this it is manifest how to a given straight line to apply a parallelogram which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure—namely, by applying to the given straight line a parallelogram equal to the first triangle ABD (I. 44), and having an angle equal to the given angle.

PROPOSITION B. PROBLEM.

To describe a triangle that shall be equal to a given rectilineal figure.

Given the figure ABCDE, *it is required* to construct a triangle equal to it.

(*Const.*) Produce the side AB both ways towards F and G; join D, B, and from C draw CG parallel to DB, and join D, G. Also, join A, D, and through E draw EF parallel to AD, and join D, F. FDG is the triangle required.



(*Dem.*) For, since DB and GC are parallel, the triangles DGB and DCB are equal (I. 37). For a similar reason, the triangles AED and AFD are equal; therefore the triangles AFD and BGD are together equal to AED and BCD; and adding ADB to these equals, the whole triangle FDG is equal to the given figure ABCDE.

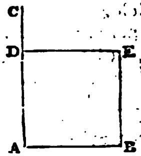
Schol.—By means of this problem any crooked boundary may be rectified; that is, a straight line may be found that will cut off the same space on each side of it that the crooked boundary does. By means of the parallel ruler, this straight boundary may be easily found. The problem is of great utility in land-surveying.

PROPOSITION XLVI. PROBLEM.

To describe a square upon a given straight line.

Given the straight line AB; *it is required* to describe a square upon AB.

(*Const.*) From the point A draw (I. 11) AC at right angles to AB, and make (I. 3) AD equal to AB, and (I. 31) through the point D draw DE parallel to AB, and through B draw BE parallel to AD; (*Dem.*) then ADEB is a parallelogram; whence AB is equal to DE (I. 34), and AD to BE. But BA is equal to AD; therefore the four straight lines BA, AD, DE, and EB are equal to one another, and the parallelogram ADEB is equilateral. It is likewise rectangular; for the straight line AD meeting the parallels AB, DE, makes the angles BAD, ADE equal to two right angles (I. 29). But BAD is a right angle; therefore also ADE is a right angle. Now, the opposite angles of parallelograms are equal (I. 34); therefore each of the opposite angles ABE and BED is a right angle; wherefore the figure ADEB is rectangular, and it has been demonstrated that it is equilateral; it is therefore a square, and it is described upon the given straight line AB.



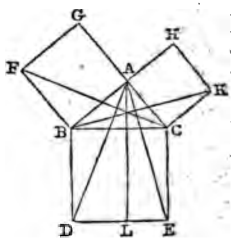
COR.—Hence every parallelogram that has one right angle has all its angles right angles.

PROPOSITION XLVII. THEOREM.

In any right-angled triangle, the square which is described upon the side opposite to the right angle, is equal to the squares described upon the sides which contain the right angle.

Given a right-angled triangle ABC, having the right angle BAC; *to prove that* the square described upon the side BC is equal to the sum of the squares described upon BA and AC.

(*Const.*) On BC describe (I. 46) the square BDEC, and on BA and AC, the squares GB and HC; and (I. 31) through A draw AL parallel to BD or CE, and join AD, FC.



(*Dem.*) Then, because each of the angles BAC and BAG is a right angle (Def. 40), the two straight lines AC and AG are in the same straight line (I. 14); for the same reason, AB and AH are in the same straight line; and because the angle DBC is equal to the angle FBA, each of them being a right angle, add to each the angle ABC, and the whole angle ABD is equal to the whole angle FBC (Ax. 2); and, because the two sides AB and BD are equal to the two FB and BC, each to each, and the angle ABD equal to the angle FBC, therefore (I. 4) the base AD is equal to the

base FC, and the triangle ABD to the triangle FBC. Now, the parallelogram BL is double of the triangle ABD (I. 41), because they are upon the same base BD, and between the same parallels BD, AL; and the square GB is double of the triangle FBC, because these also are upon the same base FB, and between the same parallels FB, GC. But the doubles of equals are equal to one another (Ax. 6); therefore the parallelogram BL is equal to the square GB. And, in the same manner, by joining AE and BK, it is demonstrated that the parallelogram CL is equal to the square HC. Therefore the whole square BDEC is equal to the two squares GB and HC; and the square BDEC is described upon the straight line BC, and the squares GB and HC upon BA and AC; wherefore the square upon the side BC is equal to the two squares upon the sides BA and AC.

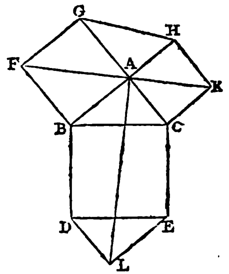
Cor.—The square on either of the sides of a right-angled triangle, is equal to the difference between the squares on the hypotenuse and the other side.

For $BC^2 = AB^2 + AC^2$, and taking AC^2 from both these equals, there remains $BC^2 - AC^2 = AB^2$; or taking AB^2 from both, then $BC^2 - AB^2 = AC^2$.

Or calling the hypotenuse, the base, and the perpendicular, H, B, and P, respectively, $H^2 = B^2 + P^2$. Therefore $H^2 - B^2 = P^2$; and $H^2 - P^2 = B^2$.

Otherwise:

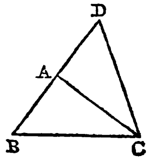
A very simple demonstration of this important proposition may be given as follows: Describe the squares GB, BE, and CH on the three sides AB, BC, and CA, and join GH, FA, and AK; describe the triangle LED, having the side LE equal to AB, and LD equal to AC, and join AL, then since DE is equal to BC (I. 34), the triangle LDE is equal to the triangle ACB in every respect (I. 8). The angles BAC and CAH being each right angles, BA and AH are in the same straight line (I. 14); for a like reason, CA and AG are in the same straight line. Again, FAK is a straight line, for GA and AF are equal to BA and AF, and the base GF is equal to the base BF, therefore the angle GAF is equal to the angle BAF (I. 8); but GAB is a right angle, therefore GAF and BAF are each equal to half a right angle; for a like reason, each of the angles HAK and CAK is half a right angle, and hence the two angles



BAF and BAK are together equal to two right, and therefore FA and AK are in the same straight line (I. 14). It is also evident that the triangles GAH and ELD are each equal in every respect to the triangle BAC. Since the triangle FBA is equal to the triangle FGA, and the triangle BAC is equal to the triangle GAH, and also the triangle CAK is equal to the triangle HAK, the whole figure FBCK is equal to the whole figure FGHK; but the figure FBCK is equal to each of the figures ABDL and LECA, for if the figure FBCK be conceived to be turned round the point C through a right angle, so that CK may coincide with CA, the point K will coincide with A, because CK is equal to CA, also BC will coincide with CE, and the point B with the point E; and since the angle CBF is evidently equal to the angle CEL, the line BF will coincide with EL, and they are equal by construction since each is equal to AB; hence the point F will coincide with L, and the line FK with the line LA; hence the figure FBCK coincides with the figure LECA, and is therefore equal to it. In the same manner, if the figure FBCK be conceived to be turned round the point B through a right angle, it may be shewn that it will coincide with the figure ABDL, and be equal to it; hence the two figures ABDLEC and FBCKHG are equal, being each double of the figure FBCK; but if from the first the triangles BAC and ELD be taken, there remains the square BE, and if from the second there be taken the two equal triangles BAC and GAH, there remains the two squares GB and HC; but BE is the square on BC, and GB and HC are the squares on AB and AC, therefore the square on BC is equal to the squares on BA and AC.

PROPOSITION XLVIII. THEOREM.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.



Given the square described upon BC, one of the sides of the triangle ABC, equal to the squares upon the other sides BA and AC; *to prove that* the angle BAC is a right angle.

(*Const.*) From the point A draw (I. 11) AD at right angles to AC, and make AD equal to BA, and join DC. (*Dem.*) Then, because DA is equal to AB, the square on DA is equal to the square on AB. To each of these add the square on AC; therefore the squares on DA and AC are equal to the squares on BA and AC. But the square on

DC is equal to the squares on DA and AC (I. 47), because DAC is a right angle; and the square on BC is given equal to the squares on BA and AC; therefore the square on DC is equal to the square on BC; and therefore also the side DC is equal to the side BC. And because the side DA is equal to AB, and AC common to the two triangles DAC and BAC, and the base DC likewise equal to the base BC, the angle DAC is equal to the angle BAC (I. 8). But DAC is a right angle; therefore also BAC is a right angle.

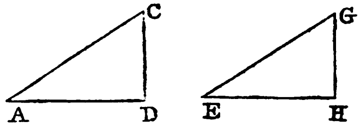
COR.—If the sides of a triangle are 3, 4, and 5, taken from any scale of equal parts, it is a right-angled triangle.

$$\text{For } 5^2 = 3^2 + 4^2, \text{ or } 25 = 9 + 16.$$

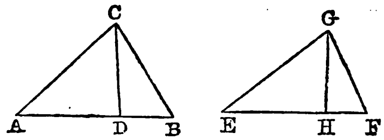
PROPOSITION C. THEOREM.

If two triangles have two sides and an angle opposite to one of them in the one, equal respectively to two sides and the corresponding angle of the other, and if the other two angles opposite to the other two equal sides are of the same species, then the triangles are equal in every respect.

First, *Given* the triangles ACD and EGH, having the sides AC and CD respectively equal to EG and GH, the angles at D and H right angles, and consequently (I. 32, Cor. 4, and Def. 15) the angles A and E of the same species; *to prove that* the base AD is equal to the base EH, the angle A equal to E, and the angle C equal to G. (*Dem.*) Since AC is $=$ EG, and CD $=$ GH, therefore $AC^2 - CD^2 = EG^2 - GH^2$; but (I. 47) $AD^2 = AC^2 - CD^2$, and $EH^2 = EG^2 - GH$; hence $AD^2 = EH^2$, and therefore AD $=$ EH. Hence the three sides of triangle ACD are respectively equal to the three sides of the triangle EGH, they are therefore every way equal (I. 8); hence, angle A $=$ angle E, and angle C $=$ angle G.

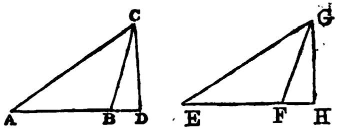


Secondly, *Given* the two triangles ABC and EFG, having the sides AC and CB respectively equal to EG and GF, and the angles BAC and FEG equal and both acute, and the angles ABC and EFG opposite to the other



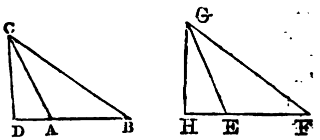
two equal sides, both acute or both obtuse; *to prove that* the triangles are equal in every other respect.

(*Const.*) Draw CD, GH, perpendicular to the base, or the base produced. (*Dem.*) Then in the triangles ACD and EGH the right angles at D and H are equal, and also angles at A and E and the sides AC and EG; they are therefore every way equal (I. 26); consequently the sides CD and GH are equal, and AD = EH. Again, the triangles BCD and FGH have two sides BC and CD in the one respectively equal to FG and GH in the other, and the angles at D and H are right angles; wherefore (by the first case) they are every way equal, and hence DB = HF. Hence the sum of AD and DB in the first two triangles, is equal to the sum of EH and HF, or AB = EF; and their differences in the second pair of triangles are also equal, or AB = EF. Hence the two triangles ABC and EFG have the three sides of the one respectively equal to the three sides of the other, and they are therefore equal in every other respect (I. 8). Hence angle ABC = EFG, and angle ACB = EGF.



Lastly, when the equal angles BAC and FEG are obtuse.

In this case, as in the preceding pair of triangles, the perpendiculars CD and GH fall without the triangles; and since the angles CAD and GEH, adjacent to the equal angles are equal (I. 13, Cor.), and the angles at D and H are right angles; hence the triangles ACD and EGH can be proved equal as in the second case; the triangles BCD and FGH can be proved equal also, as in that case, and hence AD = EH, and BD = FH; and consequently BD − AD = FH − EH, or AB = EF. Hence (I. 8) the triangles ABC and EFG are equal in every respect.



Cor.—If any two sides of one right-angled triangle are respectively equal to two corresponding sides of another, the triangles are every way equal.

EXERCISES.

1. A line* that bisects the vertical angle of an isosceles triangle, also bisects the base perpendicularly.

2. If a line be bisected perpendicularly by another line, every point in the latter is equally distant from the extremities of the former; and any point not situated in the latter is at unequal distances from the extremities of the former.

3. In a given straight line to find a point equally distant from two given points.

4. If any line be drawn through the middle point of the line joining two given points, any two points in the former line that are equidistant from the middle point are also equidistant from the two given points.

5. Of all lines that can be drawn from a given point to a given line, the perpendicular upon it is the least; and of all others, that which is nearer to the perpendicular is less than one more remote; and only two equal lines can be drawn to it from that point, one upon each side of the perpendicular.

6. If from every point of a given line, the lines drawn to each of two given points on opposite sides of the line are equal; prove that the straight line joining the given points will be bisected by the given line at right angles.

7. In the figure to Euc. I. 5, if BG and CF meet in O, shew that OA bisects the angle BAC.

8. The difference between two sides of a triangle is less than the third side.

9. If two isosceles triangles be constructed on opposite sides of the same base, the line joining their vertices bisects the common base, and each of the vertical angles.

10. Every point in the line that bisects a given angle is equidistant from the sides of the angle.

11. If the alternate extremities of two equal and parallel lines be joined, the connecting lines bisect each other.

12. If the vertical angle of an isosceles triangle be a right angle, each of the angles at the base is half a right angle.

13. If a side of an isosceles triangle be produced beyond the vertex, the exterior angle is double of either of the angles at the base.

14. The middle point of the hypotenuse of a right-angled triangle, is equally distant from each of the three angles.

* When the word line is used, straight line is understood, unless otherwise expressed.

15. Of all triangles, having the same vertical angle, and whose bases pass through a given point, the least is that whose base is bisected in that point.

16. If two sides of a triangle be produced, the lines that bisect the two exterior angles, and the angle contained by the two sides produced, pass through the same point.

17. If a right-angled triangle have one of the acute angles double of the other, prove that the hypotenuse is double of the side opposite the least angle.

18. If the exterior angle and one of the opposite interior angles, in one triangle, be respectively double those of another, the remaining opposite interior angle of the former is double that of the latter.

Note.—This proposition is the geometrical principle on which the construction of Hadley's quadrant depends.

19. Through a given point to draw a line such that the segment of it intercepted between two given parallels may be equal to a given line.

20. Through a given point to draw a line that shall be equally inclined to two given lines.

21. If two intersecting lines be respectively parallel, or equally inclined, to other two intersecting lines, the inclination of the former is equal to that of the latter.

22. The sum of two sides of a triangle is greater than twice the line joining the vertex and the middle of the base.

23. Given the sum or difference of the hypotenuse and a side of a right-angled triangle, and also the remaining side, to construct it.

24. Through a given point, between two given lines, to draw a line so that the part of it intercepted between them may be bisected in that point.

25. If two lines bisect the angles at the base of a triangle, the line joining their point of intersection and the vertex bisects the vertical angle.

26. If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

27. The sum of the perpendiculars drawn from any point within an equilateral triangle on the sides, is equal to the perpendicular from any of the angular points upon the opposite side.

28. If the diagonals of a parallelogram be equal, prove that it is a rectangle.

29. The quadrilateral figure whose diagonals bisect each other is a parallelogram.

30. The sum of the diagonals of any quadrilateral is less than the sum of any four lines that can be drawn to the four corners from any point which is not the intersection of the diagonals.

31. Half the base of a triangle is greater than, equal to, or less than, the line joining the middle of the base and the vertex, according as the vertical angle is obtuse, right, or acute.

32. If two lines bisect perpendicularly two sides of a triangle, the perpendicular, from their point of intersection upon the base, will bisect it.

33. The angle contained by a line drawn from the vertex of a triangle perpendicular to the base, and another bisecting the vertical angle, is equal to half the difference of the angles at the base.

34. To find a point in a given line such, that lines drawn from it to two given points on the same side of the given line, will make equal angles with the given line.

35. Given the sum of the sides of a triangle and the angles at the base, to construct it.

36. If two lines be drawn from the extremities of the base of a triangle to bisect the opposite sides, the line joining their intersection with the vertex, if produced, will bisect the base.

37. If in the figure to (I. 47) DB be produced to meet a perpendicular from F in N, and EC to meet a perpendicular from K in M, and if AL meet BC in O, prove that FN and KM are each equal to AO, that BN is equal to BO, and CM to CO; and that the two triangles BNF and CMK are together equal to the triangle ABC.

38. If in the figure to (I. 47) DF, GH, and KE be joined, prove that the area of the six-sided figure so formed is equal to twice the square on BC, and four times the triangle ABC.

39. If the two sides that contain the right angle in a right-angled triangle be 160 and 120, prove that the hypotenuse is 200.

40. If the hypotenuse of a right-angled triangle be 169, and one side be 65; prove that the other side is 156.

41. The angles of a quadrilateral are equal to four right angles.

42. Every quadrilateral that has its opposite sides, or opposite angles equal, is a parallelogram.

43. To find a line whose square shall be equal to the difference between two given squares.

44. To find a line whose square shall be equal to the sum of the squares on any number of lines.

45. To bisect a parallelogram by a line drawn from a point in one of its sides.

46. To bisect a given triangle by a line drawn from a point in one of its sides.

47. If three straight lines intersecting in the same point make the six angles thus formed equal, and a point be taken in one of the angles, the sum of the perpendiculars from this point upon the sides that contain the angle, will be equal to a perpendicular from the point on the third line.

48. Given a point and three lines, two of which are parallel, to find a point in each of the parallels that shall be equidistant from the given point, and such, that the line joining them shall be parallel to the other given line.

49. A line joining the middle points of two sides of a triangle, is parallel to the base, and equal to the half of it.

50. The quadrilateral formed by joining the successive middle points of the sides of a given quadrilateral, is a parallelogram.

51. If any parallelogram be described on the base of a triangle, and other two parallelograms on its two sides, such that their sides opposite to those of the triangle shall pass through the angular points of the former, the first parallelogram shall be equal to the sum or difference of the other two, according as they both lie without the triangle, or one of them upon it.

52. If one of the oblique angles of a triangle be divided into a number of equal parts, the dividing lines will divide the opposite side unequally; the segments nearer to the perpendicular being less than the more remote.

53. Any three lines being drawn making equal angles with the three sides of any triangle towards the same parts, and meeting one another, will form a triangle equiangular to the original triangle.

SECOND BOOK.

DEFINITIONS.

1. If a straight line be divided or cut at any point, the point is called a *point of section.*

2. The distance of the point of section from the middle of the line, is called the *mean distance.*

3. If the point of section lie between the extremities of the line, it is said to be cut *internally;* and if beyond one of the extremities, it is said to be cut *externally.*

4. The distances from a point of section of a line to its extremities, are called *segments;* which are said to be *internal* or *external,* according as the line is cut internally or externally.

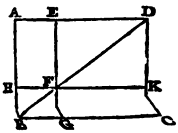
5. The rectangle *under,* or *contained by,* two lines, is a rectangle, of which these lines, or lines equal to them, are two adjacent sides.

The rectangle under, or contained by, two lines, as AB and BC, is sometimes concisely expressed thus, $AB \cdot BC$; or if A and B denote the lines, by $A \cdot B$. So the rectangle under, or contained by, a line, which is the sum of two lines A and B, and a third line C, is expressed thus $(A + B)\,C$; that under, or contained by, the excess of A above B, and another line C, by $(A - B)\,C$; and that under a line equal to $A + B$, and another equal to $A - B$, by $(A + B)(A - B)$.

6. When a line is cut into two segments, so that the rectangle under the whole line and one of the segments is equal to the square of the other segment, it is said to be cut *medially,* or in *medial section.*

7. Any of the parallelograms about a diagonal of any parallelogram, together with the two complements, is called a *gnomon.*

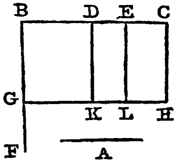
Thus the parallelogram HG, together with the complements AF, FC, is the gnomon, which is more briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



PROPOSITION I. THEOREM.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Given the two straight lines A and BC, and let BC be divided into any parts in the points D, E; *to prove that* the rectangle contained by the straight lines A and BC, is equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC.

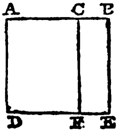


(*Const.*) From the point B draw (I. 11) BF at right angles to BC, and make BG equal to A (I. 3); and through G draw GH parallel to BC (I. 31); and through D, E, and C draw (I. 31) DK, EL, and CH parallel to BG. (*Dem.*) Then the rectangle BH is equal to the rectangles BK, DL, and EH; and BH is contained by A and BC, for it is contained by GB and BC, and GB is equal to A; and BK is contained by A and BD, for it is contained by GB and BD, of which GB is equal to A; and DL is contained by A and DE, because DK, that is, BG (I. 34), is equal to A; and in like manner the rectangle EH is contained by A and EC; therefore the rectangle contained by A and BC, is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.

PROPOSITION II. THEOREM.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of its parts are together equal to the square on the whole line.

Given the straight line AB divided into any two parts in the point C; *to prove that* the rectangle contained by AB and BC, together with the rectangle contained by AB and AC, shall be equal to the square on AB.



(*Const.*) Upon AB describe (I. 46) the square ADEB; and through C draw CF parallel to AD or BE (I. 31); (*Dem.*) then AE is equal to the rectangles AF and CE; and AE is the square of AB; and AF is the rectangle contained by BA and AC; for it is contained by DA and AC, of which AD is equal to AB; and CE is contained by AB and BC, for BE is equal to

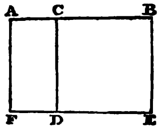
AB; therefore the rectangle contained by AB and AC, together with the rectangle AB · BC, is equal to the square on AB.

PROPOSITION III. THEOREM.

If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Given the straight line AB divided into two parts in the point C; *to prove that* the rectangle AB · BC is equal to the square on BC, together with the rectangle AC · CB.

(*Const.*) Upon BC describe (I. 46) the square CDEB, and produce ED to F, and through A draw AF parallel to CD or BE (I. 31). (*Dem.*) Then the rectangle AE is equal to the rectangles CE and AD. Now, AE is the rectangle contained by AB and BC, for it is contained by AB and BE, of which BE is equal to BC; and CE is the square on BC; and AD is the rectangle contained by AC and CB, for CD is equal to CB; therefore the rectangle AB · BC is equal to the square on BC, together with the rectangle AC · CB.

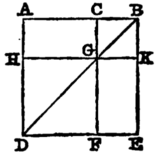


PROPOSITION IV. THEOREM.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

Given the straight line AB divided into any two parts in C; *to prove that* the square on AB is equal to the squares on AC and CB, together with twice the rectangle contained by AC and CB.

(*Const.*) Upon AB describe the square ADEB (I. 46), and join BD, and through C draw CGF parallel to AD or BE (I. 31), and through G draw HK parallel to AB or DE. (*Dem.*) And because CF is parallel to AD, and BD falls upon them, the exterior angle CGB is equal to the interior and opposite angle ADB (I. 29); but ADB is equal to the angle ABD (I. 5), because BA is equal to AD, being sides of a square; wherefore the angle CGB is equal to the angle CBG; and therefore the side BC is equal to the side CG (I. 6); but CB is equal also to GK (I. 34), and CG to BK; wherefore the figure CGKB is equilateral. It is likewise rectangular; for the angle CBK



being a right angle, the other angles of the parallelogram CGKB are also right angles (I. 46, Cor.); wherefore CGKB is a square, and it is upon the side CB. For a like reason, HF is also a square, and it is upon the side HG, which is equal to AC; therefore HF and CK are the squares on AC and CB. And because the complement AG is equal (I. 43) to the complement GE, and that AG is the rectangle contained by AC and CG, that is, by AC and CB; GE is also equal to the rectangle AC · CB; wherefore AG, GE, are together equal to twice the rectangle AC · CB. And HF, CK are the squares on AC and CB; wherefore the four figures HF, CK, AG, and GE are equal to the squares on AC and CB, and to twice the rectangle AC · CB. But HF, CK, AG, and GE make up the whole figure ADEB, which is the square on AB; therefore the square on AB is equal to the squares on AC and CB, and twice the rectangle AC · CB.

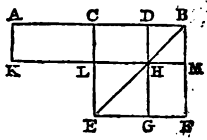
Cor.—From the demonstration, it is manifest that the parallelograms about the diagonal of a square are likewise squares.

PROPOSITION V. THEOREM.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Given the straight line AB, divided into two equal parts in the point C, and into two unequal parts at the point D; *to prove that* the rectangle AD · DB, together with the square on CD, is equal to the square on CB.

(*Const.*) Upon CB describe the square CEFB (I. 46); join BE, and through D draw DHG parallel to CE or BF (I. 31); and through H draw KLM parallel to CB or EF; and also through A draw AK parallel to CL or BM. (*Dem.*) And because the complement CH is equal to the complement HF (I. 43), to each of these add DM; therefore the whole CM is equal to the whole DF; but CM is equal to AL (I. 36), because AC is equal to CB; therefore also AL is equal to DF. To each of these add CH, and the whole AH is equal to DF and CH; but AH is the rectangle contained by AD and DB, for DH is equal to DB (II. 4, Cor.); and DF, together with CH, is the gnomon CMG; therefore the gnomon CMG is equal to the rectangle AD · DB. To each of these add LG, which is equal to the square on CD; therefore the gnomon CMG, together with LG, is equal to the rectangle



AD · DB, together with the square on CD. But the gnomon CMG and LG make up the whole figures CEFB, which is the square on CB; therefore the rectangle AD · DB, together with the square on CD, is equal to the square on CB.

COR.—From this proposition, it is manifest that the difference of the squares of two unequal lines BC, CD, is equal to the rectangle contained by their sum and difference.

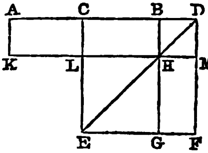
For $BC^2 = AD \cdot DB + CD^2$, and taking CD^2 from both these equals, there remains $BC^2 - CD^2 = AD \cdot DB = (BC + CD)(BC - CD)$, for $BC + CD = AC + CD = AD$, and $BC - CD = DB$.

PROPOSITION VI. THEOREM.

If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced, and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Given the straight line AB, bisected in C, and produced to the point D; *to prove that* the rectangle AD · DB, together with the square on CB, is equal to the square on CD.

(*Const.*) Upon CD describe the square CEFD (I. 46); join DE, and through B draw BHG parallel to CE or DF (I. 31), and through H draw KLM parallel to AD or EF, and also through A draw AK parallel to CL or DM. (*Dem.*) And because AC is equal to CB, the rectangle AL is equal to CH (I. 36); but CH is equal to HF (I. 43); therefore also AL is equal to HF. To each of these add CM; therefore the whole AM is equal to the gnomon CMG.



Now, AM is the rectangle contained by AD and DB, for DM is equal to DB (II. 4, Cor.); therefore the gnomon CMG is equal to the rectangle AD · DB; add to each of these LG, which is equal to the square on CB; therefore the rectangle AD · DB, together with the square on CB, is equal to the gnomon CMG, together with LG. But the gnomon CMG, together with LG, makes up the whole figure CEFD, which is the square on CD; therefore the rectangle AD · DB, together with the square on CB, is equal to the square on CD.

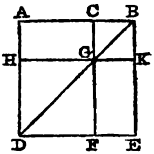
Schol.—The Corollary to Proposition 5 follows in a similar manner from this one. These two propositions are equivalent to the following: If a line be bisected, and be also cut unequally, either internally or externally, the rectangle under the unequal segments is equal to the difference between the squares on half the line and the mean distance.

PROPOSITION VII. THEOREM.

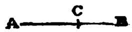
If a straight line be divided into any two parts, the squares on the whole line, and on one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Given the straight line AB, divided into any two parts in the point C; *to prove that* the squares on AB and BC are equal to twice the rectangle AB · BC, together with the square on AC.

(*Const.*) Upon AB describe the square ADEB (I. 46), and construct the figure as in the preceding propositions. (*Dem.*) And because AG is equal to GE (I. 43), add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK and CE are double of AK. But AK and CE are the gnomon AKF, together with the square CK; therefore the gnomon AKF, together with the square CK, is double of AK. But twice the rectangle AB · BC is also double of AK, for BK is equal to BC (II. 4, Cor.); therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB · BC. To each of these equals add HF, which is equal to the square on AC; therefore the gnomon AKF, together with the squares CK, HF, is equal to twice the rectangle AB · BC, and the square on AC. Now, the gnomon AKF, together with the squares CK, HF, makes up the whole figure ADEB and CK, which are the squares on AB and BC; therefore the squares on AB and BC are equal to twice the rectangle AB · BC, together with the square on AC.



Otherwise: because the square on AB is equal to the squares on AC and CB, together with twice the rectangle AC · CB; if the square on CB be added to both, the squares on AB and CB are equal to the square on AC, together with twice the square of CB, and twice the rectangle AC · CB. But the square on CB, together with the rectangle AC · CB, is equal to the rectangle AB · BC (II. 3); and therefore twice the square on CB, together with twice the rectangle AC · CB, is equal to twice the rectangle AB · BC. Wherefore the squares on AB and CB are equal to twice the rectangle AB · BC, together with the square on AC.



Cor. 1.—Hence, the sum of the squares on any two lines is equal to twice the rectangle contained by the lines, together with the square on the difference of the lines.

Cor. 2.—The square on the difference of two lines is less than the sum of their squares, by twice their rectangle.

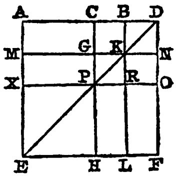
For if AB be one line and BC another, AC is their difference, and it was proved in the proposition that twice the rectangle AB · BC, together with the square on AC, is equal to the squares on AB and BC; take twice the rectangle AB · BC from each of these equals, and there remains the square on AC equal to the squares on AB and BC, diminished by twice the rectangle AB · BC.

PROPOSITION VIII. THEOREM.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first mentioned part.

Given the straight line AB, divided into any two parts in the point C; *to prove that* four times the rectangle AB · BC, together with the square on AC, is equal to the square on the straight line, which is made up of AB and BC together.

(*Const.*) Produce AB to D, so that BD be equal to CB, and upon AD describe the square AEFD; and construct two figures such as in the preceding. (*Dem.*) Because CB is equal to BD, and that CB is equal to GK (I. 34), and BD to KN; therefore GK is equal to KN. For the same reason, PR is equal to RO; and because CB is equal to BD, and GK to KN, the rectangle CK is equal to BN (I. 36), and GR to RN. But CK is equal to RN (I. 43), because they are the complements of the parallelogram CO;



therefore, also, BN is equal to GR; and the four rectangles BN, CK, GR, and RN are therefore equal to one another, and so are quadruple of one of them, CK. Again, because CB is equal to BD, and BD equal to BK (II. 4, Cor.), that is, to CG, CG is equal to CB; and CB is equal to GK, that is, to GP; therefore CG is equal to GP; and because CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to RF. But MP is equal to PL (I. 43), because they are the complements of the parallelogram ML; wherefore AG is equal also to RF; therefore the four rectangles AG, MP, PL, and RF are equal to one another, and so are quadruple of one of them, AG. And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK; therefore the eight rectangles which make up the gnomon AOH, are quadruple of AK. And because AK is the rectangle contained by AB and BC, for BK is equal to BC, four times the rectangle AB · BC is quadruple of AK. But the gnomon AOH was demonstrated

to be quadruple of AK; therefore four times the rectangle AB · BC is equal to the gnomon AOH. To each of these add XH, which is equal to the square on AC (II. 4, Cor.); therefore four times the rectangle AB · BC, together with the square on AC, is equal to the gnomon AOH and the square XH.

But the gnomon AOH and the square XH make up the figure AEFD, which is the square on AD; therefore four times the rectangle AB · BC, together with the square on AC, is equal to the square on AD; that is, on the line made up of AB and BC added together.

Cor. 1.—Hence, because AD is the sum, and AC the difference of the lines AB and BC, four times the rectangle contained by any two lines, together with the square on their difference, is equal to the square on the sum of the lines.

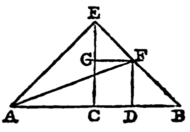
Cor. 2.—From the demonstration, it is manifest that, since the square on CD is quadruple of the square on CB, the square on any line is quadruple of the square on half that line.

PROPOSITION IX. THEOREM.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Given the straight line AB, divided at the point C into two equal, and at D into two unequal parts; *to prove that* the squares on AD and DB are together double of the squares on AC and CD.

(*Const.*) From the point C draw CE at right angles to AB (I. 11), and make it equal to AC or CB, and join EA, EB; through D draw DF parallel to CE (I. 31), and through F draw FG parallel to AB; and join AF. (*Dem.*) Then, because AC is equal to CE, the angle EAC is equal to the angle AEC (I. 5); and because the angle ACE is a right angle, the two others, AEC, EAC, together make one right angle (I. 32);



and they are equal to one another; therefore each of them is half of a right angle. For the same reason, each of the angles CEB, EBC, is half a right angle; and therefore the whole AEB is a right angle. And because the angle GEF is half a right angle, and EGF a right angle, for it is equal to the interior and opposite angle ECB (I. 29), the remaining angle EFG is half a right angle (I. 32); therefore the angle GEF is equal to the angle EFG, and the side EG equal to the side GF (I. 6). Again, because the angle at B is half a right angle,

and FDB a right angle, for it is equal to the interior and opposite angle ECB, ∴ the remaining angle BFD is half a right angle (I. 32); ∴ therefore the angle at B is equal to the angle BFD, and the side DF to the side DB. Now, because AC is equal to CE, the square on AC is equal to the square on CE; ∴ therefore the squares on AC and CE are double of the square on AC. But the square on EA is equal to the squares on AC and CE, because ACE is a right angle (I. 47); ∴ therefore the square on EA is double of the square on AC. Again, because EG is equal to GF, the square on EG is equal to the square on GF; therefore the squares on EG and GF are double of the square on GF; but the square on EF is equal to the squares on EG and GF; therefore the square on EF is double of the square on GF; and GF is equal to CD (I. 34); therefore the square on EF is double of the square on CD. But the square on AE is likewise double of the square on AC; therefore the squares on AE and EF are double of the squares on AC and CD. And the square on AF is equal to the squares on AE and EF (I. 47), because AEF is a right angle; therefore the square on AF is double of the squares on AC and CD.

But the squares on AD and DF are equal to the square on AF, because the angle ADF is a right angle; therefore, the squares on AD and DF are double of the squares on AC and CD. And DF is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

Otherwise:

Consider AC as one line, and CD as another, then AD is equal to their sum, and DB is equal to their difference; and

$$AD^2 = AC^2 + CD^2 + 2AC \cdot CD \text{ (II. 4)}, \quad \ldots \quad \ldots \quad (A)$$

$$BD^2 = AC^2 + CD^2 - 2AC \cdot CD \text{ (II. 7, Cor. 2)}, \quad \ldots \quad (B)$$

then by adding the two lines above, the result is

$$(1)\ AD^2 + DB^2 = 2AC^2 + 2CD^2 \quad \ldots \quad \ldots \quad (A) + (B)$$

which proves the proposition. Subtracting (B) from (A),

$$(2)\ AD^2 - DB^2 = 4AC \cdot CD \quad \ldots \quad \ldots \quad (A) - (B)$$

Cor. 1.—From (1). The square on the sum, together with the square on the difference of two lines, is equal to twice the sum of the squares on the lines.

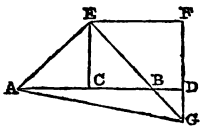
Cor. 2.—From (2). The square on the sum of two lines diminished by the square on the difference of the lines, is equal to four times the rectangle contained by the lines.

PROPOSITION X. THEOREM.

If a straight line be bisected, and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Given the straight line AB, bisected in C, and produced to the point D; *to prove that* the squares on AD and DB are double of the squares on AC and CD.

(*Const.*) From the point C draw CE at right angles to AB (I. 11); and make it equal to AC or CB, and join AE, EB; through E draw EF parallel to AB (I. 31), and through D draw DF parallel to CE. And because the straight line EF meets the parallels EC, FD, the angles CEF and EFD are together equal to two right angles (I. 29); and therefore the angles BEF, EFD are less than two right angles. But straight lines, which with another straight line make the interior angles upon the same side less than two right angles, do meet if produced far enough (I. 29, Cor.); therefore EB and FD shall meet, if produced towards B and D; let them meet in G, and join AG. (*Dem.*) Then, because AC is equal to CE, the angle CEA is equal to the angle EAC (I. 5); and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a right angle (I. 32). For the same reason, each of the angles CEB and EBC is half a right angle; therefore the angle AEB is a right angle. And because EBC is half a right angle, DBG is also half a right angle (I. 15), for they are vertically opposite; but BDG is a right angle, because it is equal to the alternate angle DCE (I. 29); therefore the remaining angle DGB is half a right angle (I. 32), and is therefore equal to the angle DBG; wherefore also the side DB is equal to the side DG (I. 6). Again, because EGF is half a right angle, and that the angle at F is a right angle, because it is equal to the opposite angle ECD (I. 34), the remaining angle FEG is half a right angle, and equal to the angle EGF; wherefore also the side GF is equal to the side FE. And because EC is equal to CA, the square on EC is equal to the square on CA; therefore the squares on EC and CA are double of the square on CA. But the square on EA is equal to the squares on EC and CA (I. 47); therefore the square on EA is double of the square on AC. Again, because GF is equal to FE, the square on GF is equal to the square on FE; and therefore the squares on GF and FE are double of the square on EF. But the square



on EG is equal to the squares on GF and FE (I. 47); therefore the square on EG is double of the square on EF; and EF is equal to CD; wherefore the square on EG is double of the square on CD; but it was demonstrated that the square on EA is double of the square on AC; therefore the squares on AE and EG are double of the squares on AC and CD. And the square on AG is equal to the squares on AE and EG; therefore the square on AG is double of the squares on AC and CD. But the squares on AD and DG are equal to the square on AG; therefore the squares on AD and DG are double of the squares on AC and CD. But DG is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

Otherwise: The same as the 'Otherwise' given in last proposition.

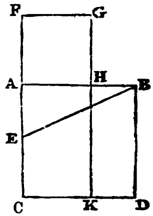
Schol.—The last two propositions are equivalent to the following: If a straight line be bisected, and be also cut unequally, either internally or externally, the sum of the squares on the two unequal segments is equal to twice the squares on half the line and on the mean distance.

PROPOSITION XI. PROBLEM.

To divide a given straight line in medial section.

Given a straight line AB; *it is required* to divide it in medial section.

(*Const.*) Upon AB describe the square ACDB (I. 46); bisect AC in E (I. 10), and join BE; produce CA to F, and make EF equal to EB (I. 3), and upon AF describe the square FAHG, and produce GH to K; AB is divided in H, so that the rectangle AB · BH is equal to the square of AH.



(*Dem.*) Because the straight line AC is bisected in E, and produced to the point F, the rectangle CF · FA, together with the square on AE, is equal to the square on EF (II. 6). But EF is equal to EB; therefore the rectangle CF · FA, together with the square on AE, is equal to the square on EB; and the squares on BA and AE are equal to the square on EB (I. 47), because the angle EAB is a right angle; therefore the rectangle CF · FA, together with the square on AE, is equal to the squares on BA and AE; take away the square on AE, which is common to both; therefore the remaining rectangle CF · FA is equal to the square on AB.* Now the figure FK is the rectangle contained by CF and FA, for AF is equal to FG, and AD is the square on AB; therefore FK is equal to AD. Take away the common part AK, and the remainder FH is equal to the remainder

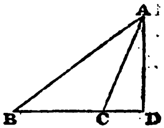
HD. And HD is the rectangle contained by AB and BH, for AB is equal to BD, and FH is the square on AH; therefore the rectangle AB · BH is equal to the square on AH. Wherefore the straight line AB is divided in H, so that the rectangle AB · BH is equal to the square on AH.

* Cor.—The line CF is also divided medially in A; but AC is equal to AB and AF to AH; hence, the line made up of the whole and its greater segment is also divided in medial section.

PROPOSITION XII. THEOREM.

In obtuse-angled triangles, if a perpendicular be drawn from any of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.

Given an obtuse-angled triangle ABC, having the obtuse angle ACB, and from the point A let AD be drawn perpendicular to BC produced (I. 12); *to prove that* the square on AB is greater than the squares on AC and CB, by twice the rectangle BC · CD.



(*Dem.*) Because the straight line BD is divided into two parts in the point C, the square on BD is equal to the squares on BC and CD, and twice the rectangle BC · CD (II. 4); to each of these equals add the square on DA; and the squares on DB and DA are equal to the squares on BC, CD, and DA, and twice the rectangle BC · CD.

But the square on BA is equal to the squares on BD and DA (I. 47), because the angle at D is a right angle; and the square on CA is equal to the squares on CD and DA. Therefore the square on BA is equal to the squares on BC and CA, and twice the rectangle BC · CD; that is, the square on BA is greater than the squares on BC and CA, by twice the rectangle BC · CD.

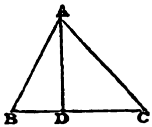
PROPOSITION XIII. THEOREM.

In every triangle, the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall upon it from the opposite angle, and the acute angle.

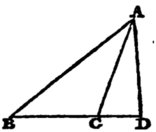
Given any triangle ABC, and the angle at B one of its acute angles, and upon BC, one of the sides containing it, let fall the

perpendicular AD from the opposite angle (I. 12); *to prove that* the square on AC, opposite to the angle B, is less than the squares on CB and BA, by twice the rectangle CB · BD.

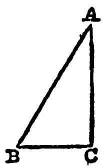
(*Dem.*) First, Let AD fall within the triangle ABC; and because the straight line CB is divided into two parts in the point D, the squares on CB and BD are equal to twice the rectangle contained by CB, BD, and the square on DC (II. 7). To each of these equals add the square on AD; therefore the squares on CB, BD, and DA are equal to twice the rectangle CB · BD, and the squares on AD and DC. But the square on AB is equal to the squares on BD and DA (I. 47), because the angle BDA is a right angle; and the square on AC is equal to the squares on AD and DC. Therefore the squares on CB and BA are equal to the square on AC, and twice the rectangle CB · BD; that is, the square on AC alone is less than the squares on CB and BA, by twice the rectangle CB · BD.



Secondly, Let AD fall without the triangle ABC; then, because the angle at D is a right angle, the angle ACB is greater than a right angle (I. 16); and therefore the square on AB is equal to the squares on AC and CB, and twice the rectangle BC · CD (II. 12). To these equals add the square on BC; therefore the squares on AB and BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC · CD. But because BD is divided into two parts in C, the rectangle DB · BC is equal to the rectangle BC · CD, and the square on BC (II. 3); and the doubles of these are equal; therefore the squares on AB and BC are equal to the square on AC, and twice the rectangle DB · BC; therefore the square on AC alone is less than the squares on AB and BC, by twice the rectangle DB · BC.



Lastly, Let the side AC be perpendicular to BC; then is BC the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares on AB and BC are equal to the square on AC, and twice the square on BC (I. 47).



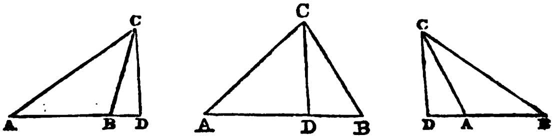
Otherwise:

PROPOSITIONS XII. AND XIII. THEOREM.

In any triangle, if a perpendicular be let fall from the vertex upon the base, then, according as the angle opposite to one of the

sides is obtuse or acute, the square on that side is greater or less than the squares on the base and the other side, by twice the rectangle contained by the base and the part of it intercepted between that angle and the perpendicular.

Given any triangle ABC, and CD a perpendicular from C upon AB or AB produced; *to prove that* $AC^2 = AB^2 + BC^2 \pm 2AB \cdot BD$, taking the upper sign when the angle ABC is obtuse, and the lower when it is acute.



$$\text{For } AD^2 = AB^2 + BD^2 \pm 2AB \cdot BD \text{ (II. 4, and 7 Cor. 2),}$$
add DC^2 to both,
$$\text{then } AD^2 + DC^2 = AB^2 + DB^2 + DC^2 \pm 2AB \cdot BD.$$
$$\text{But (I. 47) } AD_2 + DC_2 = AC^2, \quad \text{and } BD^2 + DC^2 = BC^2,$$
$$\text{therefore } AC^2 = AB^2 + BC^2 \pm 2AB \cdot BD.$$

In the demonstration + refers to the first diagram, and − to the second and third.

Schol.—If the sides opposite the angles A, B, and C be called a, b, and c, the result becomes,

$$b^2 = c^2 + a^2 \pm 2c \times BD,$$

from which we obtain in the first diagram, where the angle B is obtuse,

$$BD = \frac{b^2 - a^2 - c^2}{2c};$$

and from the second and third diagrams, where the angle B is acute,

$$BD = \frac{a^2 + c^2 - b^2}{2c}.$$

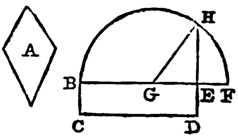
From these formulæ, the segments of the sides made by a perpendicular from any of the angles may readily be calculated when the three sides are known.

PROPOSITION XIV. PROBLEM.

To describe a square that shall be equal to a given rectilineal figure.

Given the rectilineal figure A; *it is required* to describe a square that shall be equal to A.

(*Const.*) Describe (I. 45) the rectangular parallelogram BCDE equal to the rectilineal figure A. If, then, the sides of it BE, ED are equal to one another, it is a square, and what was required is done; but if they are not equal, produce one of them BE to F, and make EF equal to ED, and bisect BF in G; and from the centre G, at the distance GB or GF, describe the semicircle BHF, and produce DE to H, and join GH.

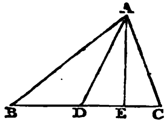


(*Dem.*) Therefore, because the straight line BF is divided into two equal parts in the point G, and into two unequal in the point E, the rectangle BE · EF, together with the square on EG, is equal to the square on GF (II. 5). But GF is equal to GH; therefore the rectangle BE · EF, together with the square on EG, is equal to the square on GH. But the squares on HE and EG are equal to the square on GH (I. 47); therefore the rectangle BE · EF, together with the square on EG, is equal to the squares on HE and EG. Take away the square on EG, which is common to both, and the remaining rectangle BE · EF is equal to the square on EH; but the rectangle contained by BE · EF is the parallelogram BD, because EF is equal to ED; therefore BD is equal to the square on EH. But BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the square on EH; wherefore the side of a square has been found equal to the given rectilineal figure A, and the square described upon EH is therefore equal to A.

PROPOSITION A. THEOREM.

If one side of a triangle be bisected, the sum of the squares on the other two sides is double of the square on half the side bisected, and of the square on the line drawn from the point of bisection to the opposite angle of the triangle.

Given a triangle ABC, of which the side BC is bisected in D, and DA drawn to the opposite angle; *to prove that* the squares on BA and AC are together double of the squares on BD and DA.



(*Const.*) From A draw AE perpendicular to BC. (*Dem.*) Because BEA is a right angle, the square on AB is equal to the squares on BE and EA (I. 47); for the same reason, the square on AC is equal to the two squares on CE and EA; therefore the squares on BA and AC are equal to the squares on BE and EC, together with twice the square on EA. But because

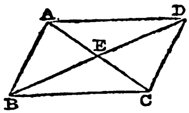
the line BC is cut equally in D, and unequally in E, the squares on BE and EC are equal to twice the squares on BD and DE (II. 9); therefore the squares on BA and AC are equal to twice the square on BD, together with twice the squares on DE and EA. Now, the squares on DE and EA are equal to the square on DA (I. 47), and therefore twice the squares on DE and EA is equal to twice the square on DA; wherefore, also, the squares on BA and AC are equal to twice the square on BD, together with twice the square on DA.

PROPOSITION B. THEOREM.

The sum of the squares on the diameters of any parallelogram is equal to the sum of the squares on the sides of the parallelogram.

Given a parallelogram ABCD, of which the diameters are AC and BD; *to prove that* the sum of the squares on AC and BD is equal to the sum of the squares on AB, BC, CD, and DA.

(*Dem.*) Let AC and BD intersect one another in E; and because the vertical angles AED, CEB are equal (I. 15), and also the alternate angles EAD, ECB (I. 29), the triangles ADE, CEB, have two angles in the one equal to two angles in the other, each to each; but the sides AD and BC, which are opposite to equal angles in these triangles, are also equal (I. 34); therefore the other sides which are opposite to the equal angles are equal (I. 26); namely, AE to EC, and ED to EB.



Since, therefore, BD is bisected in E, the squares on BA and AD are equal to twice the square on BE, together with twice the square on EA (II. A); and for the same reason the squares on BC and CD are equal to twice the square on BE, together with twice the square on EC; that is, on EA, because EC is equal to EA; therefore the four squares on BA, AD, DC, and CB are equal to four times the squares on BE and EA. But the square on BD is equal to four times the square on BE, because BD is double of BE (II. 8, Cor. 2); and for the same reason, the square on AC is equal to four times the square on AE; wherefore, also, the squares on BD and AC are equal to the four squares on BA, AD, DC, and CB.

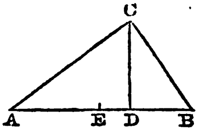
COR.—From the demonstration, it is manifest that the diameters of every parallelogram bisect one another.

PROPOSITION C. THEOREM.

The difference between the squares on the two sides of a triangle, is equal to the difference between the squares on the

segments of the base, formed by a perpendicular upon it from the vertex.

Given a triangle ABC, and CD a perpendicular on the base AB; *to prove that* $AC^2 - BC^2 = AD^2 - DB^2$.



(*Dem.*) For (I. 47) $AC^2 = AD^2 + DC^2$ and $BC^2 = DB^2 + DC^2$; therefore the difference between AC^2 and BC^2 is equal to that between $AD^2 + DC^2$ and $DB^2 + DC^2$; or $AC^2 - BC^2 = AD^2 - DB^2$.

Cor. 1.—The rectangle under the sum and difference of the two sides of a triangle, is equal to twice the rectangle under the base and the distance of its middle point from the perpendicular on it from the vertex.

Let AB be bisected in E, then $(AC + CB)(AC - CB) = 2AB \cdot ED$.

(*Dem.*) For $AC^2 - CB^2 = AD^2 - DB^2$; therefore (II. 5, Cor.) $(AC + CB)(AC - CB) = (AD + DB)(AD - DB) = AB \cdot 2ED = 2AB \cdot ED$; because $AD = AE + ED = BE + ED = DB + 2ED$, therefore $AD - DB = 2ED$, and the rectangle under AB and 2ED is equal to twice that under AB and ED.

Cor. 2.—If from any point in a given line a perpendicular be drawn to it, and from any points in the latter, lines be drawn to the extremities of the former, the differences between the squares on every two lines from the same point are equal.

For these differences are all equal to the difference between the squares on the segments of the given line.

Cor. 3.—If the difference between the squares on two lines joining a point and the extremities of a given line be equal to that on other two lines similarly drawn from another point, the line joining these two points is perpendicular to the given line, if the points be on the same side of a line bisecting the given line perpendicularly.

For the segments of the given line, formed by perpendiculars upon it from these two points, must be equal (Cor. 2), or the two perpendiculars must coincide.

Schol.—The first corollary to this proposition gives the most simple method of finding the segments of the side of a triangle made by a perpendicular upon it from the opposite angle, when the three sides are given; for by it ED may be found, and AD is equal to half the base, together with ED, while BD is equal to half the base diminished by ED, if the angle at B be acute, but if the angle at B be obtuse, BD is equal to

ED diminished by half the base. Hence, also, if ED be less than half the base, each of the angles at the base is acute; but if ED be greater than half the base, one of the angles at the base is obtuse. If the sides of the triangle opposite the angles A, B, and C be called a, b, and c; the corollary gives

$$ED = \frac{(b + a)(b - a)}{2c}.$$

Practical Example.—Let $AB = 14 = c$, $AC = 15 = b$, and $BC = 13 = a$; then

$$ED = \frac{(15 + 13)(15 - 13)}{2 \times 14} = \frac{28 \times 2}{28} = 2;$$

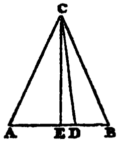
hence $AD = 7 + 2 = 9$, and $BD = 7 - 2 = 5$.

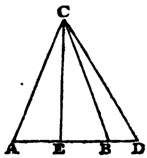
AD and BD may also be found by (II. 12 and 13), but the calculation above is much simpler.

PROPOSITION D. THEOREM.

If from the vertex of an isosceles triangle a line be drawn cutting the base, either internally or externally, the difference between the squares on this line and either side is equal to the rectangle under the segments of the base.

Given an isosceles triangle ABC, and CD a line from the vertex cutting the base; *to prove that* (1st figure) $AC^2 - CD^2 = AD \cdot DB$, and (2d fig.) $CD^2 - AC^2 = AD \cdot DB$.





For (1st fig.) $AC^2 - CD^2 = AE^2 - ED^2$ (II. c) $= (AE + ED)(AE - ED)$ (II. 5, Cor.) $= AD \cdot DB$, for $AE - ED = BE - ED = DB$.

And (2d fig.), $CD^2 - AC^2 = ED^2 - AE^2$ (II. c), $= (ED + AE)(ED - AE)$ (II. 5, Cor.) $= AD \cdot DB$, for $ED - AE = ED - EB = DB$.

COR. 1.—If + refer to the 1st figure, and − to the 2d, $AC^2 = CD^2 \pm AD \cdot DB$.

For, in the 1st figure, $AC^2 - CD^2 = AD \cdot DB$; therefore, adding to these equals CD^2, $AC^2 = CD^2 + AD \cdot DB$. And in 2d fig. $CD^2 - AC^2 = AD \cdot DB$, and adding to these equals AC^2, and taking from them $AD \cdot DB$; therefore $AC^2 = CD^2 - AD \cdot DB$.

COR. 2.—Half the difference of two unequal lines added to half their sum, gives the greater; and taken from half the sum, gives the less.

For AD, DB (1st fig.) are two unequal lines, AE is half their sum, and ED half their difference; and AE + ED = AD, and AE − ED = DB.

EXERCISES.

1. If from any point lines be drawn to the angular points of a rectangle, the sums of the squares on those drawn to opposite angles are equal.

2. The square on either of the sides of the right angle of a right-angled triangle is equal to the rectangle under the sum and difference of the hypotenuse and the other side.

3. The square on a perpendicular upon the hypotenuse of a right-angled triangle drawn from the opposite angle, is equal to the rectangle under the segments of the hypotenuse.

4. If a line be drawn from one of the acute angles of a right-angled triangle to the middle of the opposite side, the square on the hypotenuse is greater than the square on that line by three times the square on half the side bisected.

5. If from the point of bisection of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the square on the third side is equal to the difference of the squares on the segments of the hypotenuse so formed.

6. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on the hypotenuse and the segment adjacent to it, are together equal to the squares on the line so drawn, and the side on which it is drawn.

7. If a perpendicular be drawn from the right angle of a right-angled triangle on the hypotenuse, prove that the square on each of the sides is equal to the rectangle contained by the hypotenuse and its adjacent segment, and that the square on the perpendicular is equal to the rectangle contained by the segments of the hypotenuse.

8. Given two unequal lines, it is required to produce the less, so that the rectangle contained by the line thus produced and the line itself shall be equal to the square on the greater.

9. Given two straight lines, it is required to produce one of them, so that the rectangle contained by it and the produced part may be equal to the square on the other.

10. Three times the sum of the squares on the sides of a triangle, is equal to four times the sum of the squares on the three lines drawn from the angles to the middle of the opposite sides.

11. The three lines drawn from the angles of a triangle to the middle of the opposite sides all pass through the same point, and three times the sum of the squares on the lines intercepted between this point and the angles is equal to the sum of the squares on the three sides of the triangle.

12. If the three sides of a triangle be 4, 8, and 10, find by Proposition c, Scholium, whether the triangle is acute or obtuse-angled.

13. If perpendiculars be drawn from the three angles of a triangle to the opposite sides, prove that the sum of the squares on the alternate segments of the sides are equal to one another.

14. If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, twice the rectangle contained by that side and the segment of it, between the base and perpendicular, is equal to the square on the base.

15. The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equal to four times the area of the triangle.

16. If in the figure to (I. 47), the angular points of the squares be joined, thus forming a six-sided figure, prove that the sum of the squares on the six sides is equal to eight times the square on the hypotenuse of the original triangle.

17. The sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals, and four times the square on the line joining their middle points.

18. The sum of the squares on two opposite sides of a quadrilateral, together with four times the square on the line joining their middle points, is equal to the sum of the squares on the other two sides and on the diagonals.

19. The squares on the sum, and on the difference of two lines, are together double the squares on these lines.

20. If a line be cut in medial section, the line composed of it and its greater segment is similarly divided.

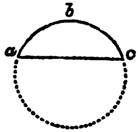
21. The sum of the squares on the diagonals of a quadrilateral, is equal to twice the sum of the squares on the lines joining the middle points of the opposite sides.

22. If the vertical angle of a triangle be four-thirds of a right angle, the square on the base is equal to the sum of the squares on the sides, together with the rectangle contained by the sides; and if the vertical angle be two-thirds of a right angle, the square on the base is equal to the sum of the squares on the two sides diminished by the rectangle contained by the sides.

THIRD BOOK.

DEFINITIONS.

1. An *arc* of a circle is any portion of the circumference, as *abc*.

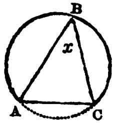


2. The *chord* of an arc is a straight line joining its extremities, as *ac*.

3. The arc of a semicircle is called a *semicircumference;* and a radius is called a *semidiameter.*

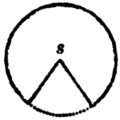
4. A *segment* of a circle is a figure contained by an arc and its chord, as S.

5. An *angle in a segment* is an angle contained by two lines, drawn from any point in its arc, to the extremities of its chord; x is the angle in the segment ABC.



6. An angle is said to *insist* or *stand* upon the arc which subtends it.

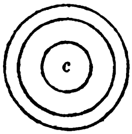
7. A *sector* of a circle is a figure contained by two radii and the intercepted arc, as *s*. When the radii are perpendicular, the sector is called a *quadrant.*



8. *Similar segments* of a circle are those that contain equal angles, as S, *s′*.

9. *Similar arcs* of circles are those that subtend equal angles at the centre.

10. *Similar sectors* are those that are bounded by similar arcs.

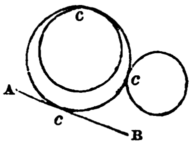


11. *Equal circles* are those that have equal radii; and *concentric* circles are those that have a common centre, *c*.

12. A straight line is said to *touch* a circle, or to be a tangent

to it, when it meets the circle, and being produced does not cut it, as AB.

13. Circles are said to *touch* one another when they meet, but do not cut one another. Such circles may be called *tangent* circles.



14. The point in which a tangent and a circle, or two tangent circles, meet, is called the *point of contact; c, c, c,* are points of contact.

15. Chords are said to be *equally distant* from the centre of a circle, when the perpendiculars upon them, from the centre, are equal; and the chord on which the greater perpendicular falls, is said to be *farther* from the centre.

16. A *secant* is a straight line drawn from a point, without a circle, and meeting it in two points. A secant may be considered a chord produced, the point without the circle from which it is drawn being a point of external section.

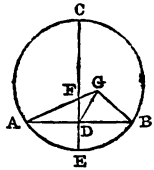
PROPOSITION I. PROBLEM.

To find the centre of a given circle.

Given the circle ABC; *it is required* to find its centre.

(*Const.*) Draw within it any chord AB, and bisect it in D (I. 10); from the point D draw DC at right angles to AB (I. 11), and produce it to E, and bisect CE in F. The point F is the centre of the circle ABC.

(*Dem.*) For, if it be not, let, if possible, G, a point not in CE, be the centre, and join GA, GD, GB. Then, because DA is equal to DB, and DG common to the two triangles ADG, BDG, the two sides AD and DG are equal to the two BD and DG, each to each; and the radius GA is equal to the radius GB; therefore the angle ADG is equal to the angle GDB (I. 8); therefore (I. Def. 11) the angle GDB is a right angle. But FDB is likewise a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the less, which is impossible; therefore G is not the centre of the circle ABC. In the same manner it can be shewn, that no point which is not in CE can be the centre; that is, the centre is in CE; and being in CE, it must be in its point of bisection; therefore F is the centre of the circle ABC.



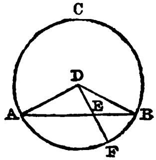
Cor.—From this it is manifest, that if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

PROPOSITION II. THEOREM.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Given a circle ABC, and A, B any two points in the circumference; *to prove that* the straight line drawn from A to B shall fall within the circle.

(*Const.*) Take any point in AB as E; find D the centre of the circle ABC; join AD, DB, and DE, and let DE meet the circumference in F. (*Dem.*) Then, because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5); and because AE, a side of the triangle DAE, is produced to B, the angle DEB is greater than the angle DAE (I. 16); but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE; DB is therefore greater than DE (I. 19). But DB is equal to DF; wherefore DF is greater than DE, and the point E is therefore within the circle. The same may be demonstrated of any other point between A and B; therefore AB is within the circle.



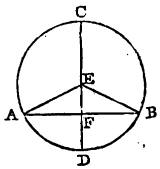
Schol.—When the two points are at the extremities of a diameter, or diametrically opposite, the truth of the proposition is evident. It is obvious, from the nature of the circle, when they are at a less distance. When the points are exceedingly near, however, it is not then so evident that the chord lies wholly within the circle, and no part of it on the arc; and therefore, in this case, it is necessary to demonstrate the proposition. Some parts of the figure, however, in this extreme case, would be so small as to be indistinct, and therefore it is necessary to assume the points at a sufficient distance; and as the reasoning in this case applies to every case, the proposition will therefore be established also in the extreme case. A similar observation applies to several propositions, which appear to be axiomatic, except in the extreme case, as I. 20.

PROPOSITION III. THEOREM.

If a straight line drawn through the centre of a circle bisect a chord in it which does not pass through the centre, it shall cut it at right angles; and if it cuts it at right angles, it shall bisect it.

Given a circle ABC, and CD, a straight line drawn through the centre, bisecting any chord AB, which does not pass through the centre, in the point F; *to prove that* it cuts it also at right angles.

(*Const.*) Take E the centre of the circle (III. 1), and join EA, EB. (*Dem.*) Then, because AF is equal to FB, and FE common to the two triangles AFE and BFE,



there are two sides in the one equal to two sides in the other, and the base EA is equal to the base EB; therefore the angle AFE is equal to the angle BFE (I. 8); therefore (I. Def. 11) each of the angles AFE and BFE is a right angle; wherefore the straight line CD, drawn through the centre bisecting another AB that does not pass through the centre, cuts the same at right angles. Second,

Given that CD cuts AB at right angles; *to prove that* CD also bisects AB; that is, that AF is equal to FB.

The same construction being made, (*Dem.*) because EA and EB from the centre are equal to one another, the angle EAF is equal to the angle EBF (I. 5); and the right angle AFE is equal to the right angle BFE; therefore, in the two triangles EAF and EBF, there are two angles in one equal to two angles in the other, and the side EF, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal (I. 26); AF is therefore equal to FB.

Cor.—If a line bisect a chord of a circle, and be perpendicular to it, it will pass through the centre.

For a line from the centre bisecting the chord is perpendicular to it, and therefore coincides with the former perpendicular; therefore it must also pass through the centre.

Schol.—It appears, therefore, that of the three conditions—of passing through the centre of a circle, of being perpendicular to a chord, and of bisecting a chord; if a line fulfil two, it will also fulfil the third.

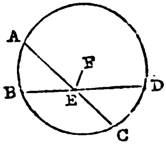
PROPOSITION IV. THEOREM.

If in a circle two chords cut one another which do not both pass through the centre, they do not bisect each other.

Given a circle ABCD, and AC, BD, two chords in it, which cut one another in the point E, and do not both pass through the centre; *to prove that* AC and BD do not bisect one another.

(*Dem.*) For, if it be possible, let AE be equal to EC, and BE to ED. If one of the lines pass through the centre, it is plain that it cannot be bisected by the other which does not pass through the centre. But if neither of them pass through the centre, take F the centre of the circle, and join EF (III. 1); and because FE, a straight line through the centre, bisects

another AC which does not pass through the centre, it shall cut it at right angles (III. 3); wherefore FEA is a right angle. Again, because the straight line FE bisects the straight line BD, which does not pass through the centre, it shall cut it at right angles; wherefore FEB is a right angle; and FEA was shewn to be a right angle; therefore FEA is equal to the angle FEB, the less to the greater, which is impossible; therefore AC and BD do not bisect one another.

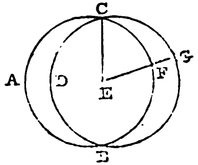


PROPOSITION V. THEOREM.

If two circles cut one another, they shall not have the same centre.

Given the two circles ABC and CDG, which cut one another in the points B, C; *to prove that* they have not the same centre.

(*Const.*) For, if it be possible, let E be their centre; join EC, and draw any straight line EFG meeting them in F and G; (*Dem.*) and because E is the centre of the circle ABC, CE is equal to EF. Again, because E is the centre of the circle CDG, CE is equal to EG; but CE was shewn to be equal to EF; therefore EF is equal to EG, the less to the greater, which is impossible; therefore E is not the centre of the circles ABC and CDG, and the same may be proved of any other point, hence they have not the same centre.

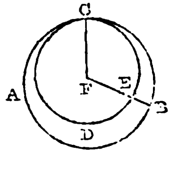


PROPOSITION VI. THEOREM.

If one circle touch another internally, the circles shall not have the same centre.

Given the two circles ABC and CDE which touch internally in the point C; *to prove that* they have not the same centre.

(*Const.*) For, if they can, let it be F; join FC, and draw any straight line FEB meeting them in E and B; (*Dem.*) and because F is the centre of the circle ABC, CF is equal to FB; also, because F is the centre of the circle CDE, CF is equal to FE; and CF was shewn equal to FB; therefore FE is equal to FB, the less to the greater, which is impossible; wherefore F is not the centre of



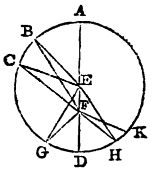
the circles ABC and CDE, and the same may be proved of any other point, hence they have not the same centre.

PROPOSITION VII. THEOREM.

If any point be taken in the diameter of a circle which is not the centre, of all the straight lines which can be drawn from it to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of the others, that which is nearer to the line which passes through the centre is always greater than one more remote. And from the same point there cannot be drawn more than two straight lines that are equal to one another, one upon each side of the diameter.

Given a circle ABCD, and AD its diameter, and E the centre, in which any point F is taken which is not the centre. *To prove that* of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA is the greatest, and FD, the other part of the diameter AD, is the least; and of the others, FB is greater than FC, and FC than FG.

(*Const.*) Join BE, CE, GE; (*Dem.*) and (I. 20) BE, EF are greater than BF; but AE is equal to EB; therefore AE, EF, that is, AF is greater than BF. Again, because BE is equal to CE, and FE common to the triangles BEF and CEF, the two sides BE and EF are equal to the two CE and EF; but the angle BEF is greater than the angle CEF; therefore the base BF is greater than the base FC (I. 24); for a like reason, CF is greater than GF. Again, because GF and FE are greater than EG, and EG is equal to ED; GF and FE are greater than ED; take away the common part FE, and the remainder GF is greater than the remainder FD; therefore FA is the greatest, and FD the least of all the straight lines from F to the circumference; and BF is greater than CF, and CF than GF.



Also, there cannot be drawn more than two equal straight lines from the point F to the circumference, one upon each side of the diameter; at the point E in the straight line EF, make the angle FEH equal to the angle GEF (I. 23), and join FH.

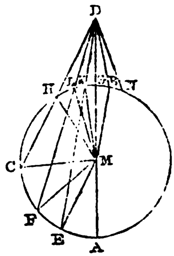
Then, because GE is equal to EH, and EF common to the two triangles GEF and HEF; the two sides GE and EF are equal to the two HE and EF; and the angle GEF is equal to the angle HEF; therefore the base FG is equal to the base FH (I. 4). But besides FH, no straight line can be drawn from F to the circumference equal to FG; for, if there can, let

it be FK; and because FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which passes through the centre, is equal to one which is more remote, which is impossible.

PROPOSITION VIII. THEOREM.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, whereof one passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre; and of the rest, that which is nearer to that through the centre is always greater than the more remote; but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than the more remote; and only two equal straight lines can be drawn from the point to the circumference, one upon each side of the least.

Given a circle ABC, and D any point without it, from which are drawn the straight lines DA, DE, DF, and DC to the circumference, whereof DA passes through the centre. *To prove that* of those which fall upon the concave part of the circumference AEFC, the greatest is AD which passes through the centre; and the nearer to it is always greater than the more remote; namely, DE greater than DF, and DF greater than DC; but of those which fall upon the convex circumference HLKG, the least is DG between the point D and the diameter AG; and the nearer to it is always less than the more remote; namely, DK less than DL, and DL less than DH.



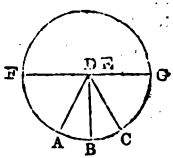
(*Const.*) Take M the centre of the circle ABC (III. 1), and join ME, MF, MC, MK, ML, and MH. (*Dem.*) And because AM is equal to ME, add MD to each; therefore AD is equal to EM and MD; but EM and MD are greater than ED (I. 20); therefore, also, AD is greater than ED. Again, because ME is equal to MF, and MD common to the triangles EMD and FMD; EM and MD are equal to FM and MD; but the angle EMD is greater than the angle FMD; therefore the base ED is greater than the base FD (I. 24). In like manner it may be shewn that FD is greater than CD; therefore DA is the greatest, and DE greater than DF, and DF than DC. And because MK, KD are greater than MD (I. 20), and MK is equal to MG, the remainder KD is greater than the remainder GD

(Ax. 5); that is, GD is less than KD. And because MK, DK are drawn to the point K within the triangle MLD from M and D, the extremities of its side MD; MK and KD are less than ML and LD (I. 21), whereof MK is equal to ML; therefore the remainder DK is less than the remainder DL. In like manner it may be shewn that DL is less than DH; therefore DG is the least, and DK less than DL, and DL than DH. Also, there can be drawn only two equal straight lines from the point D to the circumference, one upon each side of the least; for at the point M, in the straight line MD, make the angle DMB equal to the angle DMK (I. 23), and join DB; and, because in the triangles KMD and BMD, the side KM is equal to the side BM, and MD common to both, and also the angle KMD equal to the angle BMD, the base DK is equal to the base DB (I. 4). But, besides DB. no straight line can be drawn from D to the circumference, equal to DK; for, if there can, let it be DN; then, because DN is equal to DK, and DK equal to DB, DB is equal to DN; that is, the line nearer to DG, the least is equal to the more remote, which has been shewn to be impossible.

PROPOSITION IX. THEOREM.

If a point be taken within a circle, from which there fall more than two equal straight lines to the circumference, that point is the centre of the circle.

Given the point D within the circle ABC, from which to the circumference there fall more than two equal straight lines—namely, DA, DB, and DC; *to prove that* the point D is the centre of the circle.



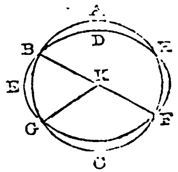
(*Dem.*) For, if not, let E be the centre; join DE, and produce it to the circumference in F and G; then FG is a diameter of the circle ABC. And because in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, DG shall be the greatest line from it to the circumference, and DC greater than DB (III. 7), and DB than DA; but they are likewise equal, which is impossible; therefore E is not the centre of the circle ABC. In like manner it may be demonstrated that no other point but D is the centre; D therefore is the centre.

Schol.—The proposition is easily proved without a figure, thus: The point from which more than two equal lines can be drawn to the circumference of a circle, cannot be an eccentric point (III. 7); therefore it must be the centre.

PROPOSITION X. THEOREM.

The circumference of a circle cannot cut that of another in more than two points.

(*Const.*) If it be possible, let the circumference FAB cut the circumference DEF in more than two points; namely, in B, G, and F; take the centre K of the circle ABC, and join KB, KG, and KF. (*Dem.*) And because K is the centre of the circle ABC, and B, G, and F are points on the circumference, the three straight lines KB, KG, and KF are all equal; and because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, and KF, the point K is the centre of the circle DEF (III. 9); but K is also the centre of the circle ABC; therefore the same point is the centre of two circles that cut one another, which is impossible (III. 5).

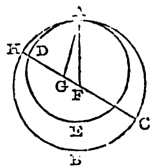


PROPOSITION XI. THEOREM.

If two circles touch each other internally, the straight line which joins their centres being produced, shall pass through the point of contact.

Given the two circles ABC, ADE touching each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE; *to prove that* the straight line which joins the centres F, G, being produced, passes through the point A.

(*Dem.*) For, if not, let it fall otherwise, if possible, as FGDH, and join AF, AG; and because AG, GF are greater than FA (I. 20); that is, than FH, for FA is equal to FH, being each a radius of the same circle; take away the common part FG, and the remainder AG will be greater than the remainder GH. But AG is equal to GD, being each a radius of the circle ADE, therefore GD is greater than GH; and it is also less, which is impossible; therefore the straight line which joins the centres cannot fall otherwise than on the point A; that is, it must pass through A.

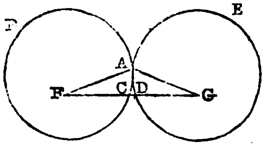


PROPOSITION XII. THEOREM.

If two circles touch each other externally, the straight line which joins their centres shall pass through the point of contact.

Given the two circles ABC, ADE touching each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE; *to prove that* the straight line which joins the points F and G shall pass through the point of contact A.

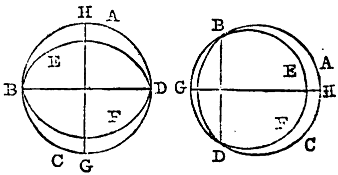
(*Const.*) For, if not, let it pass otherwise, if possible, as FCDG, and join FA, AG; (*Dem.*) and because F is the centre of the circle ABC, AF is equal to FC. Also, because G is the centre of the circle ADE, AG is equal to GD; therefore FA and AG are equal to FC and DG; wherefore the whole FG is greater than FA and AG; but it is also less (I. 20), which is impossible; therefore the straight line which joins the centres cannot pass otherwise than through the point of contact A; that is, it must pass through it.



PROPOSITION XIII. THEOREM.

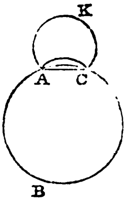
One circle cannot touch another in more points than one, whether it touches it internally or externally.

(*Const.*) For, if it be possible, let the circle EBF touch the circle ABC in more points than one, and first internally, in the points B and D; join BD, and draw GH bisecting BD at right angles (I. 10, 11); (*Dem.*) therefore, because the points B and D are in the circumference of each of the circles, the straight line BD falls within each of them (III. 2); and their centres are in the straight line GH which bisects BD at right angles (III. 1, Cor.); therefore GH, being the line joining their centres, passes through the point of contact (III. 11); but it does not pass through it, because the points B and D are without the straight line GH, which is absurd; therefore one circle cannot touch another on the inside in more points than one.



Nor can two circles touch one another externally in more than

one point; for, if it be possible, let the circle ACK touch the circle ABC in the points A and C, and join AC; therefore, because the two points A and C are in the circumference of the circle ACK, the straight line AC which joins them shall fall within the circle ACK (III. 2). And the circle ACK is without the circle ABC; and therefore the straight line AC is without this last circle; but because the points A and C are in the circumference of the circle ABC, the straight line AC must be within the same circle, which is absurd; therefore one circle cannot touch another on the outside in more than one point; and it has been shewn that they cannot touch on the inside in more points than one.

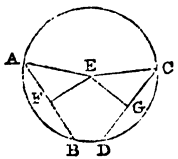


PROPOSITION XIV. THEOREM.

Equal chords in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

Given the chords AB and CD, in the circle ABDC, equal to one another; *to prove that* they are equally distant from the centre.

(*Const.*) Take E the centre of the circle ABDC, and from it draw EF and EG, perpendiculars to AB and CD; (*Dem.*) then, because the straight line EF, passing through the centre, cuts the chord AB, which does not pass through the centre, at right angles, it also bisects it (III. 3); wherefore AF is equal to FB, and AB double of AF. For the same reason, CD is double of CG; and AB is equal to CD; therefore AF is equal to CG. And because AE is equal to EC, the square on AE is equal to the square on EC. But the squares on AF and FE are equal to the square on AE (I. 47), because the angle AFE is a right angle; and, for a like reason, the squares on EG and GC are equal to the square on EC; therefore the squares on AF and FE are equal to the squares on CG and GE, of which the square on AF is equal to the square on CG, because AF is equal to CG; therefore the remaining square on FE is equal to the remaining square on EG, and the straight line EF is therefore equal to EG; therefore (III. Def. 15) AB and CD are equally distant from the centre.



Next, let it be given that FE is equal to EG; *to prove that* AB is equal to CD. (*Dem.*) For, the same construction being made, it may, as before, be demonstrated, that AB is

double of AF, and CD double of CG, and that the squares on EF and FA are equal to the squares on EG and GC; of which the square on FE is equal to the square on EG, because FE is equal to EG; therefore the remaining square on AF is equal to the remaining square on CG; and the straight line AF is therefore equal to CG. And AB is double of AF, and CD double of CG; wherefore AB is equal to CD.

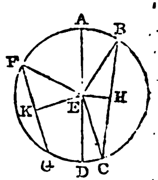
Schol.—A principle employed in this and the next proposition is—if two quantities A and B be together equal to other two C and D together, that is, if $A + B = C + D$; then, if $A = C$, $B = D$; or, if $A > C$, $B < D$.

PROPOSITION XV. THEOREM.

The diameter is the greatest chord in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Given a circle ABCD, of which the diameter is AD, and the centre E; and BC a chord nearer to the centre than FG; *to prove that* AD is greater than any chord BC which is not a diameter, and BC, which is nearer the centre, is greater than FG, which is more remote.

(*Const.*) From the centre draw EH and EK, perpendiculars to BC and FG, and join EB, EC, and EF; (*Dem.*) and because AE is equal to EB, and ED to EC, AD is equal to EB and EC. But EB and EC are greater than BC (I. 20); wherefore, also, AD is greater than BC.



And because BC is nearer to the centre than FG, EH is less than EK (III. Def. 15). But, as was demonstrated in the preceding, BC is double of BH, and FG double of FK, and the squares on EH and HB are equal to the squares on EK and KF, of which the square on EH is less than the square on EK, because EH is less than EK; therefore the square on BH is greater than the square on FK, and the straight line BH greater than FK; and therefore BC is greater than FG.

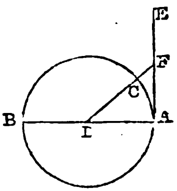
Next, let it be given that BC is greater than FG; *to prove that* BC is nearer to the centre than FG; that is, the same construction being made, EH is less than EK; (*Dem.*) because BC is greater than FG, BH likewise is greater than KF; and the squares on BH and HE are equal to the squares on FK and KE, of which the square on BH is greater than the square on FK, because BH is greater than FK; therefore the square

on EH is less than the square on EK, hence the straight line EH is less than EK.

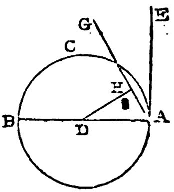
PROPOSITION XVI. THEOREM.

The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; (2) and a straight line making an acute angle with the diameter at its extremity cuts the circle.

Given a circle ABC, the centre of which is D, and the diameter AB; and AE a line drawn from A perpendicular to AB; *to prove that* AE shall fall without the circle.



(*Const.*) In AE take any point F; join DF, and let DF meet the circle in C. (*Dem.*) Because DAF is a right angle, it is greater than the angle AFD (I. 32); therefore DF is greater than DA (I. 19); and DA is equal to DC; therefore DF is greater than DC, and the point F is therefore without the circle. Now, F is any point whatever in the line AE; therefore AE falls without the circle.



2. (*Const.*) Again, let AG be drawn, making the angle DAG less than a right angle; from D draw DH at right angles to AG; (*Dem.*) and because the angle DHA is a right angle, each of the other angles of the triangle DAH is less than a right angle; the angle DAH is therefore less than the angle DHA, and therefore also the side DH is less than the side DA. Hence the point H is within the circle, and therefore the straight line AG cuts the circle.

COR.—From this it is manifest that the straight line which is drawn at right angles to the radius of a circle from the extremity of it, is a tangent; and that it touches it only in one point, because if it did meet the circle in two, it would fall within it (III. 2). Also, it is evident that there can be but one straight line which touches the circle in the same point.

PROPOSITION XVII. PROBLEM.

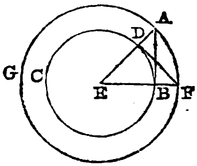
To draw a straight line from a given point, either without or in the circumference, which shall be a tangent to a given circle.

First, *Given* the circle BCD and a point A without it; it is

required to draw a straight line from A which shall be a tangent to the circle.

(*Const.*) Find the centre E of the circle (III. 1), and join AE; and from the centre E, at the distance EA, describe the circle AFG; from the point D draw DF at right angles to EA (I. 11), and join EF and AB. AB touches the circle BCD.

(*Dem.*) Because E is the centre of the circles BCD and AFG; EA is equal to EF, and ED to EB; therefore the two sides AE and EB are equal to the two FE and ED, and they contain the angle at E common to the two triangles AEB and FED; therefore the base DF is equal to the base AB, and the triangle FED to the triangle AEB, and the other angles to the other angles (I. 4); therefore the angle EBA is equal to the angle EDF. But EDF is a right angle, wherefore EBA is a right angle; and EB is a radius; but a straight line drawn from the extremity of a radius, at right angles to it, is a tangent (III. 16, Cor.); therefore AB touches the circle, and it is drawn from the given point A, which was required to be done.

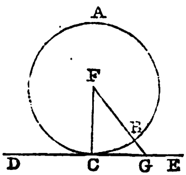


Second, If the given point be in the circumference of the circle, as the point D, draw the radius DE, and draw DF at right angles to DE; DF touches the circle (III. 16, Cor.).

PROPOSITION XVIII. THEOREM.

If a straight line be a tangent to a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the tangent.

Given the straight line DE a tangent to the circle ABC at the point C; (*Const.*) take the centre F, and draw the straight line FC; it is required *to prove that* FC is perpendicular to DE.

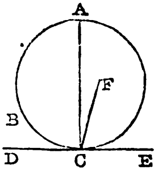


(*Dem.*) For, if it be not, from the point F draw FBG perpendicular to DE; and because FGC is a right angle, GCF is an acute angle (I. 17); therefore (I. 19) FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, and it is also less, which is impossible; wherefore FG is not perpendicular to DE. In the same manner it may be shewn, that no other is perpendicular to it besides FC; that is, FC is perpendicular to DE.

PROPOSITION XIX. THEOREM.

If a straight line be a tangent to a circle, and from the point of contact a straight line be drawn perpendicular to the tangent, the centre of the circle shall be in that line.

Given that the straight line DE is a tangent to the circle ABC, in C, and from C let CA be drawn at right angles to DE; *to prove that* the centre of the circle is in CA.



(*Const.*) For, if not, let, if possible, F be the centre, and join CF. (*Dem.*) Because DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular to DE (III. 18); therefore FCE is a right angle. But ACE is also a right angle; therefore the angle FCE is equal to the angle ACE, the less to the greater, which is impossible; wherefore F is not the centre of the circle ABC. In the same manner, it may be shewn, that no other point which is not in CA, is the centre; that is, the centre is in CA.

Cor.—If a line, drawn from the centre of a circle, be perpendicular to a tangent, it will pass through the point of contact.

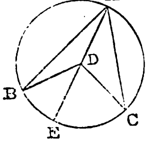
For the line drawn from the centre to the point of contact is perpendicular to the tangent, and it must therefore coincide with the former line, which therefore must pass through the point of contact.

Schol.—From the last two propositions and corollary, it appears that, of the three conditions—of passing through the centre of a circle, of being perpendicular to a tangent, and of passing through the point of contact—if a line fulfil two, it also fulfils the third.

PROPOSITION XX. THEOREM.

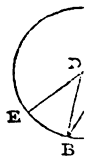
The angle at the centre of a circle is double of the angle at the circumference, upon the same base; that is, upon the same arc.

Given a circle ABC, and BDC an angle at its centre, and BAC an angle at its circumference, which have the same arc BC for their base; *to prove that* the angle BDC is double of the angle BAC.



(*Const.*) First, let D, the centre of the circle, be within the angle BAC, and join AD, and produce it to E. (*Dem.*) Because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5); therefore the angles DAB and DBA are double of the angle DAB; but (I. 32) the

angle BDE is equal to the angles DAB and DBA; t also the angle BDE is double of the angle DAB. For t reason, the angle EDC is double of the angle DAC; therefore the whole angle BDC is double of the whole angle BAC.



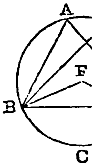
Again, let D, the centre of the circle, be without the angle BAC, and join AD and produce it to E. It may be demonstrated, as in the first case, that the angle EDG is double of the angle DAC, and that EDB, a part of the first is double of DAB, a part of the other; therefore the remaining angle BDC is d the remaining angle BAC.

*** See Appendix—Proposition (c).

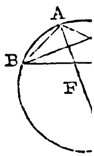
PROPOSITION XXI. THEOREM.

The angles in the same segment of a circle are equa another.

Given a circle ABCD, and BAD, BED angles in t segment BAED; *to prove that* the angles BAD and BED are equal to one another.



(*Const.*) Take F the centre of the circle ABCD; and, first, let the segment BAED be greater than a semicircle, and join BF, FD. (*Dem.*) And because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same arc—namely, BCD for their base; therefore the angle BFD is double of the angle BAD. For the same reason, the angle BFD is of the angle BED; therefore the angle BAD is equa angle BED.



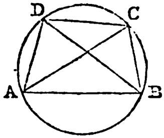
But if the segment BAED be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to one another. Draw AF to the centre and produce it to C, and join CE; therefore the segment BADC is greater than a semicircle; and the angles in it BAC and BEC are equal by the first case. For the same reason, because CBED is greater than a semicircle, the angles CAD and CED are equal; t the whole angle BAD is equal to the whole angle BED.

PROPOSITION XXII. THEOREM.

The opposite angles of any quadrilateral figure described in a circle, are together equal to two right angles.

Given ABCD a quadrilateral figure in the circle ABCD; *to prove that* any two of its opposite angles are together equal to two right angles.

(*Const.*) Join AC, BD; (*Dem.*) and because the three angles of every triangle are equal to two right angles (I. 32), the three angles of the triangle CAB—namely, the angles CAB, ABC, and BCA are equal to two right angles.

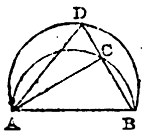


But the angle CAB is equal to the angle CDB (III. 21), because they are in the same segment BADC, and the angle ACB is equal to the angle ADB, because they are in the same segment ADCB; therefore the whole angle ADC is equal to the angles CAB and ACB. To each of these equals add the angle ABC; therefore the angles ABC, CAB, and BCA are equal to the angles ABC and ADC. But ABC, CAB, and BCA are equal to two right angles; therefore also the angles ABC and ADC are equal to two right angles. In the same manner, the angles BAD and DCB may be shewn to be equal to two right angles.

PROPOSITION XXIII. THEOREM.

Upon the same straight line and upon the same side of it, there cannot be two similar segments of circles not coinciding with one another.

(*Const.*) If it be possible, let the two similar segments of circles —namely, ACB and ADB, be upon the same side of the same straight line AB, not coinciding with one another; then, because the circle ACB cuts the circle ADB in the two points A and B, they cannot cut one another in any other point (III. 10); one of the segments must therefore fall within the other; let ACB fall within ADB, and draw the straight line BCD, and join CA, DA;



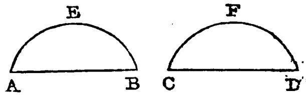
(*Dem.*) and because the segment ACB is similar to the segment ADB, the angle ACB is equal to the angle ADB (III. Def. 8), the exterior to the interior, which is impossible (I. 16); therefore on the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding.

PROPOSITION XXIV. THEOREM.

Similar segments of circles upon equal straight lines are equal to one another.

Given AEB and CFD two similar segments of circles upon the equal straight lines AB and CD; *to prove that* the segment AEB is equal to the segment CFD.

(*Dem.*) For, if the segment AEB be applied to the segment CFD, so that the point A be on C, and the straight line AB upon CD, the point B shall coincide with the point D, because AB is equal to CD; therefore the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (III. 23), and therefore is equal to it.

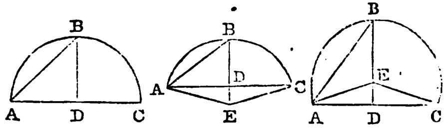


PROPOSITION XXV. PROBLEM.

A segment of a circle being given, to describe the circle of which it is the segment.

Given the segment of a circle ABC; *it is required* to describe the circle of which it is the segment.

(*Const.*) Bisect AC in D (I. 10), and from the point D draw DB at right angles to AC (I. 11), and join AB; *first*, let the angles ABD and BAD be equal to one another; then the straight line BD is equal to DA (I. 6), and therefore to DC; (*Dem.*) and because the three straight lines DA, DB, and DC are all equal, D is the centre of the circle (III. 9); from



the centre D, at the distance of any of the three DA, DB, or DC, describe a circle; this shall pass through the other points; and the circle, of which ABC is a segment, is described; and because the centre D is in AC, the segment ABC is a semicircle; *second*, but if the angles ABD and BAD are not equal to one another, (*Const.*) at the point A, in the straight line AB, make the angle BAE equal to the angle ABD (I. 23), and produce BD, if necessary, to E, and join EC; (*Dem.*) and

because the angle ABE is equal to the angle BAE, the straight line BE is equal to EA; and because AD is equal to DC, and DE common to the triangles ADE and CDE, the two sides AD and DE are equal to the two CD and DE, each to each; and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base AE is equal to the base EC (I. 4); but AE was shewn to be equal to EB, wherefore also BE is equal to EC; and the three straight lines AE, EB, and EC are therefore equal to one another; wherefore E is the centre of the circle (III. 9); and if from the centre E, at the distance of any of the three AE, EB, or EC, a circle be described, it shall pass through the other points; and be the circle, of which ABC is a segment. It is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the segment ABC, which therefore is less than a semicircle; but if the angle ABD be less than BAD, the centre E falls within the segment ABC, which is therefore greater than a semicircle; wherefore a segment of a circle being given, the circle is described of which it is a segment.

Otherwise:

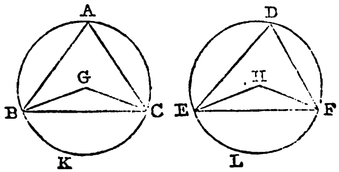
Draw any two chords in the given segment, then draw two straight lines bisecting these chords at right angles, they will intersect in the centre (III. 1); the centre being thus found, the circle can be described

PROPOSITION XXVI. THEOREM.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or at the circumferences.

Given ABC and DEF two equal circles, and the equal angles BGC and EHF at their centres, and BAC and EDF at their circumferences; *to prove that* the arc BKC is equal to the arc ELF.

(*Const.*) Join BC and EF; (*Dem.*) and because the circles ABC and DEF are equal, the straight lines drawn from their centres are equal; therefore the two sides BG and GC are equal to the two EH and HF; and the angle at G is equal to the angle at H; therefore the base BC is equal to the base EF (I. 4); and because the angle at A is equal to the angle at D, the segment BAC is similar to the segment EDF



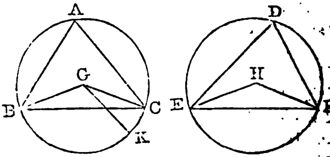
(III. Def. 8); and they are upon equal straight lines BC and EF; but similar segments of circles upon equal straight lines are equal to one another (III. 24); therefore the segment BAC is equal to the segment EDF; but the whole circle ABC is equal to the whole DEF; therefore the remaining segment BKC is equal to the remaining segment ELF, and the arc BKC is equal to the arc ELF.

PROPOSITION XXVII. THEOREM.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or at the circumferences.

Given the angles BGC and EHF at the centres, and BAC and EDF at the circumferences, of the equal circles ABC and DEF, standing upon the equal arcs BC and EF; *to prove that* the angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

(*Const.*) If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to EDF (III. 20). But if not, one of them is the greater; let BGC be the greater, and at the point G in the straight line BG, make the angle BGK equal to the angle EHF (I. 23); (*Dem.*) but equal angles stand upon equal arcs (III. 26) when they are at the centre; therefore the arc BK is equal to the arc EF; but EF is equal BC; therefore also BK is equal to BC, the less to the greater, which is impossible; therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it; and the angle at A is half of the angle BGC, and the angle at D half of the angle EHF; therefore the angle at A is equal to the angle at D.

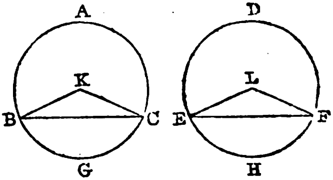


PROPOSITION XXVIII. THEOREM.

In equal circles, equal chords cut off equal arcs, the greater equal to the greater, and the less to the less.

Given ABC and DEF two equal circles, and BC and EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less BGC, EHF; *to prove that* the greater BAC is equal to the greater EDF; and the less BGC to the less EHF.

(*Const.*) Take K and L the centres of the circles (III. 1), and join BK, KC, EL, and LF; (*Dem.*) and because the circles are equal, their radii are equal; therefore BK and KC are equal to EL and LF; and the base BC is equal to the base EF; therefore the angle BKC is equal to the angle ELF (I. 8). But equal angles stand upon equal arcs (III. 26) when they are at the centres; therefore the arc BGC is equal to the arc EHF. But the whole circle ABC is equal to the whole EDF; therefore the remaining part of the circumference, namely, BAC, is equal to the remaining part EDF.

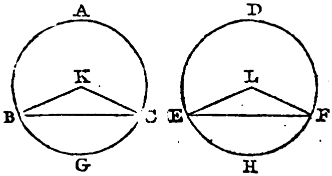


PROPOSITION XXIX. THEOREM.

In equal circles equal arcs are subtended by equal chords.

Given ABC and DEF two equal circles, and BGC and EHF equal arcs; *to prove that* the chord BC is equal to the chord EF.

(*Const.*) Join BC and EF, and take K and L the centres of the circles (III. 1), and join BK, KC, EL, and LF; (*Dem.*) and because the arc BGC is equal to the arc EHF, the angle BKC is equal to the angle ELF (III. 27); and because the circles ABC and DEF are equal, their radii are equal; therefore BK and KC are equal to EL and LF, and they contain equal angles; therefore the base BC is equal to the base EF (I. 4).



Schol.—The last four propositions are evidently true, if instead of the words 'in equal circles,' there be substituted 'in the same circle,' and they are often so applied.

PROPOSITION XXX. PROBLEM.

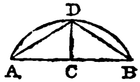
To bisect a given arc; that is, to divide it into two equal parts.

Given the arc ADC; *it is required* to bisect it.

(*Const.*) Join AB, and bisect it in C (I. 10); from the point

C draw CD at right angles to AB, and join AD and DB; the arc ADB is bisected in the point D.

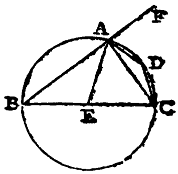
(*Dem.*) Because AC is equal to CB, and CD common to the triangles ACD and BCD, the two sides AC and CD are equal to the two BC and CD; and the angle ACD is equal to the angle BCD, because each of them is a right angle; therefore the base AD is equal to the base BD (I. 4). But equal chords cut off equal arcs (III. 28), the greater equal to the greater, and the less to the less; and AD and DB are each of them less than a semicircle, because DC passes through the centre (III. 1, Cor.); wherefore the arc AD is equal to the arc DB; therefore the given arc is bisected in D.



PROPOSITION XXXI. THEOREM.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Given a circle ABCD, of which the diameter is BC, and centre E; and draw CA dividing the circle into the segments ABC and ADC, and join BA, AD, and DC; *to prove that* the angle in the semicircle BAC is a right angle; and the angle in the segment ABC, which is greater than a semicircle, is less than a right angle; and the angle in the segment ADC, which is less than a semicircle, is greater than a right angle.



(*Const.*) Join AE, and produce BA to F; (*Dem.*) and because BE is equal to EA, the angle EAB is equal to EBA (I. 5); also, because AE is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC and ACB. But FAC, the exterior angle of the triangle ABC, is also equal to the two angles ABC and ACB; therefore the angle BAC is equal to the angle FAC, and they are adjacent angles, therefore each of them is a right angle (I. Def. 11); wherefore the angle BAC in a semicircle is a right angle.

And because the two angles ABC and BAC, of the triangle ABC, are together less than two right angles (I. 17), and that BAC is a right angle, ABC must be less than a right angle; and therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.

And because ABCD is a quadrilateral figure in a circle, any

two of its opposite angles are together equal to two right angles (III. 22); therefore the angles ABC and ADC are equal to two right angles; and ABC is less than a right angle; wherefore the other ADC is greater than a right angle.

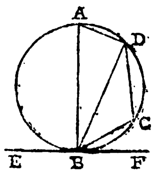
COR.—From this it is manifest, that if one angle of a triangle be equal to the other two, it is a right angle, because the angle adjacent to it is equal to the same two; and when the adjacent angles are equal, they are right angles.

PROPOSITION XXXII. THEOREM.

The angles contained by a tangent to a circle, and a chord drawn from the point of contact, are equal to the angles in the alternate segments of the circle.

Given EF a tangent to the circle ABCD, and BD a chord drawn from the point of contact B cutting the circle; *to prove that* the angles which BD makes with the tangent EF shall be equal to the angles in the alternate segments of the circle; that is, the angle FBD is equal to the angle which is in the segment DAB, and the angle DBE to the angle in the segment BCD.

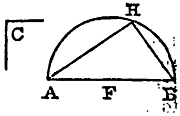
(*Const.*) From the point B draw BA at right angles to EF (I. 11), and take any point C in the circumference BD, and join AD, DC, and CB; (*Dem.*) and because the straight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the tangent from the point of contact B, the centre of the circle is in BA (III. 19); therefore the angle ADB in a semicircle is a right angle (III. 31), and consequently the other two angles BAD and ABD are equal to a right angle (I. 32); but ABF is likewise a right angle; therefore the angle ABF is equal to the angles BAD and ABD; take from these equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is the angle in the alternate segment of the circle. Again, because ABCD is a quadrilateral figure in a circle, the opposite angles BAD and BCD are equal to two right angles (III. 22); but the angles DBF and DBE are likewise equal to two right angles (I. 13); therefore the angles DBF and DBE are equal to the angles BAD and BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle.



PROPOSITION XXXIII. PROBLEM.

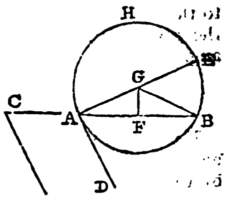
Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

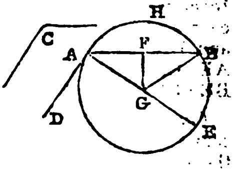
Given the straight line AB, and the rectilineal angle at C; *it is required* to describe upon the given straight line AB a segment of a circle, containing an angle equal to the angle C.



(*Const.*) First, let the angle at C be a right angle, and bisect AB in F (I. 10), and from the centre F, at the distance FB, describe the semicircle AHB; (*Dem.*) since the angle AHB is in a semicircle, it is a right angle (III. 31), and is therefore equal to the angle at C.

(*Const.*) Second, if the angle C be not a right angle, at the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23), and from the point A draw AE at right angles to AD; bisect AB in F (I. 11), and from F draw FG at right angles to AB, and join GB. (*Dem.*) And because AF is equal to FB, and FG common to the triangles AFG and BFG, the two sides AF and FG are equal to the two BF and FG; and the angle AFG is equal to the angle BFG; therefore the base AG is equal to the base GB (I. 4); and the circle described from the centre G, at the distance GA, shall pass through the point B; let this be the circle AHB. And because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE, therefore AD touches the circle (III. 16, Cor.); and because AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate segment AHB (III. 32). But the angle DAB is equal to the angle C, therefore also the angle C is equal to the angle in the segment AHB. Wherefore, upon the given straight line AB the segment AHB of a circle is described which contains an angle equal to the given angle at C.



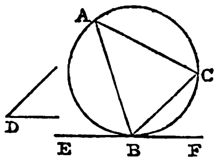


PROPOSITION XXXIV. PROBLEM.

To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.

Given the circle ABC, and the rectilineal angle D; *it is required* to cut off a segment from the circle ABC that shall contain an angle equal to the angle D.

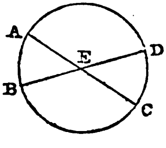
(*Const.*) Draw the straight line EF touching the circle ABC in the point B (III. 17), and at the point B, in the straight line BF, make the angle FBC equal to the angle D (I. 23); (*Dem.*) therefore, because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal to the angle BAC in the alternate segment of the circle (III. 32). But the angle FBC is equal to the angle D; therefore the angle in the segment BAC is equal to the angle D; wherefore the segment BAC is cut off from the given circle ABC containing an angle equal to the given angle D.



PROPOSITION XXXV. THEOREM.

If two chords in a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

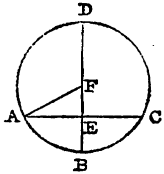
Given the two chords AC and BD, within the circle ABCD, cutting one another in the point E; *to prove that* the rectangle contained by AE and EC, is equal to the rectangle contained by BE and ED.



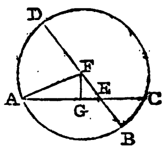
Case First.—If AC and BD pass each of them through the centre, so that E is the centre; it is evident that AE, EC, BE, and ED, being all equal, the rectangle AE · EC, is likewise equal to the rectangle BE · ED.

Case Second.—But let one of them BD pass through the centre, and cut the other AC, which does not pass through the centre, at right angles, in the point E; then if BD be bisected in F, F is the centre of the circle ABCD; join AF; (*Dem.*) and because BD, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles in E; AE and EC are equal to one another (III. 3); and because the straight line BD is cut into

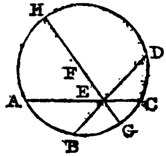
two equal parts in the point F, and into two unequal in the point E, the rectangle BE · ED, together with the square on EF, is equal to the square on FB (II. 5); that is, to the square of FA; but the squares on AE and EF are equal to the square on FA (I. 47); therefore the rectangle BE · ED, together with the square on EF, is equal to the squares on AE and EF. Take away the common square on EF, and the remaining rectangle BE · ED is equal to the remaining square on AE; that is, to the rectangle AE · EC, since AE is equal to EC.



Case Third.—Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in E, but not at right angles; then, as before, if BD be bisected in F, F is the centre of the circle. Join AF, and from F draw FG perpendicular to AC (I. 12); therefore AG is equal to GC (III. 3); (*Dem.*) wherefore the rectangle AE · EC, together with the square on EG, is equal to the square on AG (II. 5). To each of these equals add the square on GF; therefore the rectangle AE · EC, together with the squares on EG and GF, is equal to the squares on AG and GF. But the squares on EG and GF are equal to the square on EF; and the squares on AG and GF are equal to the square on AF; therefore the rectangle AE · EC, together with the square on EF, is equal to the square on AF; that is, to the square on FB. But the square on FB is equal to the rectangle BE · ED, together with the square on EF (II. 5); therefore the rectangle AE · EC, together with the square on EF, is equal to the rectangle BE · ED, together with the square on EF. Take away the common square on EF, and the remaining rectangle AE · EC is therefore equal to the remaining rectangle BE · ED.



Case Fourth.—Let neither of the straight lines AC, BD pass through the centre; take the centre F, and through E, the intersection of the straight lines AC and DB, draw the diameter GEFH; (*Dem.*) and because the rectangle GE · EH is equal, by Case Third, to the rectangle AE · EC; and also to the rectangle BE · ED; therefore the rectangle AE · EC is equal to the rectangle BE · ED.



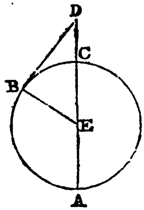
⁂ See Appendix—Proposition (D).

PROPOSITION XXXVI. THEOREM.

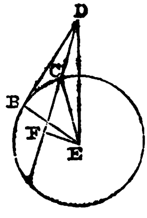
If from a point without a circle a secant and a tangent to it be drawn, the rectangle under the secant and its external segment is equal to the square on the tangent.

Given any point D without the circle ABC, and DCA and DB two straight lines drawn from it; of which DCA cuts the circle, and DB touches the same; *to prove that* the rectangle AD · DC is equal to the square on DB.

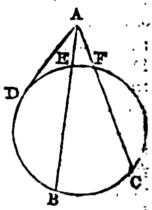
Either DCA passes through the centre, or it does not. *First*, let it pass through the centre E, and join EB; (*Dem.*) therefore the angle EBD is a right angle (III. 18); and because the straight line AC is bisected in E, and produced to the point D, the rectangle AD · DC, together with the square on EC, is equal to the square on ED (II. 6); and CE is equal to EB; therefore the rectangle AD · DC, together with the square on EB, is equal to the square on ED; but (I. 47) the square on ED is equal to the squares on EB and BD, because EBD is a right angle; therefore the rectangle AD · DC, together with the square on EB, is equal to the squares on EB and BD; take away the common square on EB; therefore the remaining rectangle AD · DC is equal to the square on the tangent DB.



Second, when DCA does not pass through the centre of the circle ABC, take the centre E (I. 3), and draw EF perpendicular to AC (I. 12), and join EB, EC, and ED; (*Dem.*) and because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it shall likewise bisect it (III. 3); therefore AF is equal to FC; and because the straight line AC is bisected in F, and produced to D, the rectangle AD · DC, together with the square on FC, is equal to the square on FD (II. 6); to each of these equals add the square on FE; therefore the rectangle AD · DC, together with the squares on CF and FE, is equal to the squares on DF and FE; but the square on ED is equal to the squares on DF and FE, because EFD is a right angle; and the square on EC is equal to the squares on CF and FE; therefore the rectangle AD · DC, together with the square on EC, is equal to the square on ED; and CE is equal to EB; therefore the rectangle AD · DC, together with the square on EB, is equal



to the square on ED; but the squares on EB and BD are also equal to the square on ED, because EBD is a right angle; therefore the rectangle AD · DC, together with the square on EB, is equal to the squares on EB and BD; take away the common square on EB; therefore the remaining rectangle AD · DC is equal to the square on DB.



COR.—If from any point without a circle there be drawn two straight lines cutting it, as AB and AC, the rectangles contained by the whole lines and the parts of them without the circle are equal to one another—namely, the rectangle BA · AE, to the rectangle CA · AF; (*Dem.*) for each of them is equal to the square of the straight line AD which touches the circle.

Schol. 1.—The above corollary and proposition 35, may be enunciated thus: The rectangles under the segments of two intersecting chords of a circle are equal, whether they cut internally or externally.

Schol. 2.—The first case of this proposition affords a geometrical principle by which the length of the diameter of the earth may be computed.

Thus, if CD (first fig.) represent the height of a mountain, which, of course, is greatly exaggerated in size compared with AC, which represents the diameter of the earth; and if this height is known, and also the distance DB of the horizon, then since $AD \cdot DC = DB^2$, it follows that the product of the numbers representing the lengths of CD and AD is equal to the square of the number denoting the length of DB; and this square number being divided by CD, gives AD for a quotient, and then AC is known. This method, however, does not afford very precise results (see PRACTICAL MATHEMATICS). The proposition is also the geometrical principle on which the method of levelling is founded.

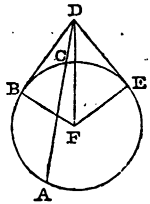
PROPOSITION XXXVII. THEOREM.

If from a point without a circle a secant be drawn, and also a line meeting the circle, and if the rectangle under the secant and its external segment be equal to the square on the other line, this line will be a tangent.

Given any point D without the circle ABC, and two straight lines DCA and DB drawn from it, of which DCA cuts the circle, and DB meets it; *to prove that* if the rectangle AD · DC be equal to the square on DB, DB touches the circle.

(*Const.*) Draw the straight line DE, touching the circle ABC

(III. 17); find the centre F, and join FE, FB, and FD; (*Dem.*) then FED is a right angle (III. 18); and because DE touches the circle ABC, and DCA cuts it, the rectangle AD · DC is equal to the square on DE (III. 36); but the rectangle AD · DC is equal to the given square on DB; therefore the square on DE is equal to the square on DB; and the straight line DE is equal to the straight line DB; and FE is equal to FB, wherefore DE and EF are equal to DB and BF; and the base FD is common to the two triangles DEF and DBF; therefore the angle DEF is equal to the angle DBF (I. 8); and DEF is a right angle; therefore also DBF is a right angle; but FB is a radius, and the straight line which is drawn at right angles to a radius, from the extremity of it, touches the circle (III. 16, Cor.); therefore DB touches the circle ABC.



COR.—Hence two tangents drawn to a circle from the same point are equal.

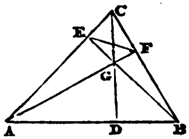
(*Dem.*) For DE and DB are two tangents, and the square on each is equal to the rectangle AD · DC; therefore $DE^2 = DB^2$, and consequently DE = DB.

PROPOSITION A. THEOREM.

If from the extremities of the base of a triangle perpendiculars be drawn to the opposite sides, the line joining their intersection and the vertex is perpendicular to the base.

Given a triangle ABC, and AF, BE perpendiculars upon the sides, intersecting in G; *to prove that* CGD is perpendicular to the base AB.

(*Const.*) Join EF. (*Dem.*) Since the quadrilateral figure CEGF has the angles at E and F right angles, a circle may be described about it (III. 22), therefore the angle GEF is equal to the angle GCF (III. 21); again, since the angles AEB and AFB are right angles, a circle described on AB as diameter would pass through E and F (III. 31), therefore the angle BEF, that is, GEF is equal to the angle BAF (III. 21), wherefore the angles GAD and GCF are equal, each being equal to GEF; but the angles AGD and CGF are also equal (I. 15), therefore the triangles AGD and CGF have two angles of the one equal to two angles of the other, hence the angle ADG is equal to the angle CFG, but



CFG is a right angle by construction, therefore ADG is a right angle, wherefore CD is perpendicular to AB.

EXERCISES.

1. If a line drawn from the centre of a circle bisect or be perpendicular to a chord, it will bisect and be perpendicular to all chords that are parallel to the former.

2. If two circles cut each other, the line joining the points of intersection is bisected perpendicularly by the line joining their centres.

3. If two circles cut one another, and from one of the points of intersection two diameters be drawn, their other extremities, and the other point of intersection, will be in one straight line.

4. A line that bisects two parallel chords in a circle, is also perpendicular to them.

5. Two concentric circles intercept between them, two equal portions of a line cutting them both.

6. Of these five conditions—of passing through the centre of a circle, bisecting a chord, being perpendicular to a chord, bisecting the subtended angle at the centre, bisecting the subtended arc of the chord—if a line fulfil any two, it will also fulfil the other three.

This exercise comprehends ten theorems, of which three are proved in III. 3, and its corollary.

7. If a circle be described on the radius of another circle, any chord in the latter, drawn from the point in which the circles meet, is bisected by the former.

8. The exterior angle of a quadrilateral figure inscribed in a circle, is equal to the interior and opposite.

9. Parallel chords in a circle intercept equal arcs.

10. If two circles touch one another, either internally or externally, the chords of the two arcs, intercepted by two lines drawn through the point of contact, are parallel.

11. If a tangent to a circle be parallel to a chord, the point of contact bisects the intercepted arc.

12. If a chord to the greater of two concentric circles be a tangent to the less, it is bisected in the point of contact.

13. If any number of circles intersect a given circle, and pass through two given points, the lines joining the intersections of each circle will all meet in the same point.

14. Through two given points, to describe a circle touching a given circle.

15. If two chords in a circle intersect each other perpendicularly, the sum of the squares on their four segments is equal to the square on the diameter.

16. Perpendiculars, from the extremities of a diameter of a circle, upon any chord, cut off equal segments.

17. Given the vertical angle, the base, and the altitude of a triangle, to construct it.

18. In a circle, the sum of the squares on two lines drawn from the extremities of a chord, to any point in a diameter parallel to it, is equal to the sum of the squares on the segments of the diameter.

19. Given the vertical angle, the base, and the sum of the sides of a triangle, to construct it.

20. If the points of contact of two tangents to a circle be joined, any secant drawn from their intersection is divided into three segments, such that the rectangle under the secant and its middle segment is equal to that under its extreme segments.

21. If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the line joining the points of concourse, is equal to the sum of the squares on the two tangents from these points.

22. Two parallel chords in a circle are respectively six and eight inches in length, and are one inch apart; find the diameter of the circle.

23. If two circles cut each other, the greatest line that can be drawn through either of the points of intersection, and is terminated by the circles, is parallel to the line joining their centres.

24. Describe a circle which shall touch a given circle in a given point, and also touch a given straight line.

25. Of all lines which touch the interior, and are bounded by the exterior of two circles which touch internally, the radius of the inner circle being greater than half that of the outer, the greatest is that which is parallel to the common tangent.

26. Shew that the two tangents to a circle drawn from the same point without it are equal to one another; and hence prove that the sums of the opposite sides of any quadrilateral described about a circle are equal, and the angles subtended at the centre of the circle by any two opposite sides are together equal to two right angles.

27. Describe a circle whose centre shall be in the base of a right-angled triangle, and which shall touch the hypotenuse and pass through the right angle.

28. If a circle be described about an equilateral triangle, and any point be taken in the circumference, the sum of the lines drawn from that point to the two adjacent angles, shall be equal to the line drawn from the same point to the opposite angle.

29. Determine a point in the diameter of a circle produced, from which a tangent being drawn to the circle, it shall be equal to the diameter.

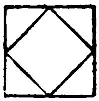
30. Describe a circle which shall touch a given circle, and also touch a given line in a given point.

31. If in the diameter or diameter produced of a circle, two points be taken equally distant from the centre; the sum of the squares on the distances of any point on the circumference from these two points is constant. Also shew that if the point taken on the circle be in the diameter, it gives the proof of the ninth or tenth proposition of the Second Book of Euclid, according as the points are in the diameter produced, or in the diameter.

32. If on the three sides of any triangle, equilateral triangles be described; straight lines joining the centres of the circles described about these three triangles, will form an equilateral triangle.

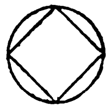
FOURTH BOOK.

DEFINITIONS.

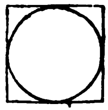


1. A rectilineal figure is said to be *inscribed in* another rectilineal figure, when all the angles of the inscribed figure are upon the sides of the figure in which it is inscribed, each upon each.

2. In like manner, a figure is said to be *described about* another figure, when all the sides of the circumscribed figure pass through the angular points of the figure about which it is described, each through each.

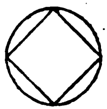


3. A rectilineal figure is said to be *inscribed in* a circle, when all the angles of the inscribed figure are upon the circumference of the circle.



4. A rectilineal figure is said to be *described about* a circle, when each side of the circumscribed figure touches the circumference of the circle.

5. In like manner, a circle is said to be *inscribed in* a rectilineal figure, when the circumference of the circle touches each side of the figure.



6. A circle is said to be *described about* a rectilineal figure, when the circumference of the circle passes through all the angular points of the figure about which it is described.

7. A straight line is said to be *placed in* a circle, when the extremities of it are in the circumference of the circle.

8. A *regular* polygon has all its sides equal, and also all its angles.

9. A polygon of five sides is called a *pentagon;* of six, a *hexagon;* of seven, a *heptagon;* of eight, an *octagon;* of nine, a *nonagon;* of ten, a *decagon;* of eleven, an *undecagon;* of twelve, a *dodecagon;* and of fifteen, a *quindecagon* or *pentedecagon.*

10. The *centre* of a regular polygon is a point equally distant from its sides or angular points.

11. The *apothem* of a regular polygon is a perpendicular from its centre upon any of its sides.

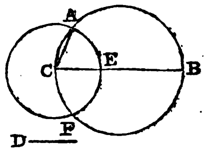
12. The *perimeter* or *periphery* of any figure is its circumference or whole boundary.

PROPOSITION I. PROBLEM.

In a given circle to place a straight line, equal to a given straight line not greater than the diameter of the circle.

Given the circle ABC, and the straight line D, not greater than the diameter of the circle; *it is required* to place in the circle ABC a straight line equal to D.

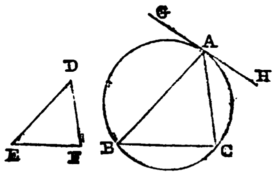
(*Const.*) Draw BC a diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a straight line BC is placed equal to D; but if it is not, BC is greater than D; make CE equal to D (I. 3), and from the centre C, at the distance CE, describe the circle AEF, and join CA; (*Dem.*) therefore, because C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA. Wherefore, in the circle ABC, a straight line is placed, equal to the given straight line D, which is not greater than the diameter of the circle.



PROPOSITION II. PROBLEM.

In a given circle to inscribe a triangle equiangular to a given triangle.

Given the circle ABC, and the triangle DEF; *it is required* to inscribe in the circle ABC a triangle equiangular to the triangle DEF.



(*Const.*) Draw the straight line GAH touching the circle in the point A (III. 17), and at the point A, in the straight line AH, make the angle HAC equal to the angle DEF (I. 23); and at the point A, in the straight line AG, make the angle GAB equal to the angle DFE, and join BC. (*Dem.*) Therefore, because HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is equal to the angle ABC in the

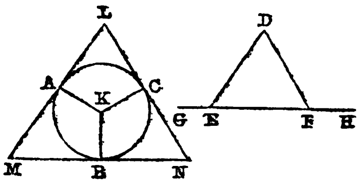
alternate segment of the circle (III. 32); but HAC is equal to the angle DEF; therefore also the angle ABC is equal to DEF; for the same reason, the angle ACB is equal to the angle DFE; therefore the remaining angle BAC is equal to the remaining angle EDF (I. 32); wherefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC.

PROPOSITION III. PROBLEM.

About a given circle to describe a triangle equiangular to a given triangle.

Given the circle ABC, and the triangle DEF; *it is required* to describe a triangle about the circle ABC equiangular to the triangle DEF.

(*Const.*) Produce EF both ways to the points G, H, and find the centre K of the circle ABC, and from it draw any straight line KB; at the point K in the straight line KB, make the angle BKA equal to the angle DEG (I. 23), and the angle BKC equal to the angle DFH; and through the points A, B, and C, draw the straight lines LAM, MBN, and NCL, touching the circle ABC (III. 17). (*Dem.*) Therefore, because LM, MN, and NL touch the circle ABC in the points A, B, and C, to which from the centre are drawn KA, KB, and KC, the angles at the points A, B, and C are right angles (III. 18); and because the four angles of the quadrilateral figure AMBK are equal to four right angles, for it can be divided into two triangles; and because two of them KAM and KBM are right angles, the other two AKB and AMB are together equal to two right angles. But the angles DEG and DEF are likewise equal to two right angles (I. 13); therefore the angles AKB and AMB are equal to the angles DEG and DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF.



In like manner, the angle LNM may be demonstrated to be equal to DFE; and therefore the remaining angle MLN is equal to the remaining angle EDF (I. 32); wherefore the triangle LMN is equiangular to the triangle DEF; and it is described about the circle ABC.

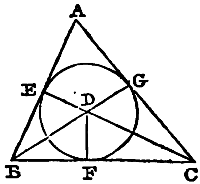
PROPOSITION IV. PROBLEM.

To inscribe a circle in a given triangle.

Given the triangle ABC; *it is required* to inscribe a circle in the triangle ABC.

(*Const.*) Bisect the angles ABC and BCA, by the straight lines BD and CD, meeting one another in the point D, from which draw DE, DF, and DG perpendiculars to AB, BC, and CA.

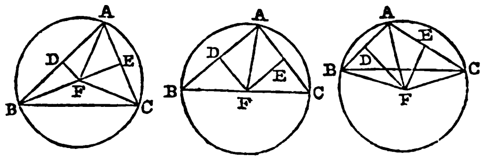
(*Dem.*) And because the angle EBD is equal to the angle FBD, the angle ABC being bisected by BD; and because the right angle BED is equal to the right angle BFD, the two triangles EBD and FBD have two angles of the one equal to two angles of the other; and the side BD, which is opposite to the right angles in each, is common to both; therefore their other sides are equal (I. 26); wherefore DE is equal to DF. For a like reason, DG is equal to DF; therefore the three straight lines DE, DF, and DG are equal to one another, and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two, and will touch the straight lines AB, BC, and CA, because the angles at the points E, F, and G are right angles, and the straight line which is drawn from the extremity of a radius at right angles to it, touches the circle (III. 16, Cor.); therefore the straight lines AB, BC, and CA do each of them touch the circle, and the circle EFG is inscribed in the triangle ABC.



PROPOSITION V. PROBLEM.

To describe a circle about a given triangle.

Given the triangle ABC; *it is required* to describe a circle about ABC.



(*Const.*) Bisect (I. 10) AB and AC, in the points D and E, and from these points draw DF and EF at right angles to AB and AC (I. 11); DF and EF produced, will meet one

another; for if DE be joined, the two angles FDE and FED are together less than two right angles (I. 29, Cor.). Let them meet in F, and join FA; also, if the point F be not in BC, join BF and CF. (*Dem.*) Then because AD is equal to DB, and DF common, and at right angles to AB, the base AF is equal to the base FB (I. 4). In like manner, it may be shewn that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, and FC are equal to one another; wherefore the circle described from the centre F, at the distance of one of them shall pass through the extremities of the other two, and be described about the triangle ABC.

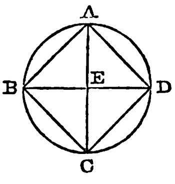
Cor.—It is manifest that when the centre of the circle falls within the triangle, each of its angles is less than a right angle, each of them being in a segment greater than a semicircle; but when the centre is in one of the sides of the triangle, the angle opposite to this side, being in a semicircle, is a right angle; and if the centre falls without the triangle, the angle opposite to the side beyond which it is, being in a segment less than a semicircle, is greater than a right angle. Wherefore, if the given triangle be acute-angled, the centre of the circle falls within it; if it be a right-angled triangle, the centre is in the hypotenuse; and if it be an obtuse-angled triangle, the centre falls without the triangle, beyond the side opposite to the obtuse angle.

PROPOSITION VI. PROBLEM.

To inscribe a square in a given circle.

Given the circle ABCD; *it is required* to inscribe a square in ABCD.

(*Const.*) Draw the diameters AC and BD at right angles to one another, and join AB, BC, CD, and DA. (*Dem.*) Because BE is equal to ED, E being the centre, and because EA is at right angles to BD, and common to the triangles ABE and ADE; the base BA is equal to the base AD (I. 4); and, for the same reason, BC and CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is also rectangular; for the straight line BD, being the diameter of the circle ABCD, BAD is a semicircle; wherefore the angle BAD is a right angle (III. 31); for the same reason, each of the angles ABC, BCD, and CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been shewn to be equilateral; therefore it is a square; and it is inscribed in the circle ABCD.



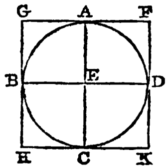
PROPOSITION VII. PROBLEM.

To describe a square about a given circle.

Given the circle ABCD; *it is required* to describe a square about it.

(*Const.*) Draw two diameters AC and BD, of the circle ABCD, at right angles to one another, and (III. 17) through the points A, B, C, and D, draw FG, GH, HK, and KF, touching the circle.

(*Dem.*) And because FG touches the circle ABCD, and EA is drawn from the centre E to the point of contact A, the angles at A are right angles (III. 18); for the same reason, the angles at the points B, C, and D are right angles; and because the angle AEB and EBG are right angles, GH is parallel to AC (I. 28); for the same reason, AC is parallel to FK, therefore GH is parallel to FK (I. 30); and in like manner, GF, HK may each of them be demonstrated to be parallel to BED, and therefore parallel to one another; hence the figures GK, GC, AK, FB, and BK are parallelograms; and GF is therefore equal to HK (I. 34), and GH to FK; and because AC is equal to BD, and that AC is equal to each of the two GH and FK; and BD to each of the two GF and HK; therefore GH, FK, GF, and HK are all equal; therefore the quadrilateral figure FGHK is equilateral. It is also rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB is likewise a right angle.



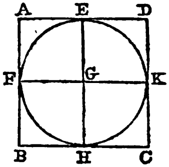
In the same manner it may be shewn that the angles at H, K, and F are right angles; therefore the quadrilateral figure FGHK is rectangular; and it was demonstrated to be equilateral; therefore it is a square; and it is described about the circle ABCD.

PROPOSITION VIII. PROBLEM.

To inscribe a circle in a given square.

Given the square ABCD; *it is required* to inscribe a circle in it.

(*Const.*) Bisect (I. 10) each of the sides AB and AD in the points F and E, and through E draw EH parallel to AB or DC (I. 31), and through F draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, and GD is a parallelogram, and their opposite sides are equal (I. 34). (*Dem.*)



And because AD is equal to AB, and that AE is the half of AD,

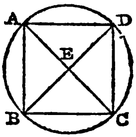
and AF the half of AB, AE is equal to AF; wherefore the sides opposite to these are equal—namely, FG to GE; in the same manner it may be demonstrated that GH and GK are each of them equal to FG or GE; therefore the four straight lines GE, GF, GH, and GK are equal to one another; and the circle described from the centre G, at the distance of one of them, shall pass through the extremities of the other three; and shall also touch the straight lines AB, BC, CD, and DA, because the angles at the points E, F, H, and K are right angles (I. 29), and because the straight line which is drawn from the extremity of a radius, at right angles to it, touches the circle (III. 16, Cor.); therefore each of the straight lines AB, BC, CD, and DA touches the circle, which therefore is inscribed in the square ABCD.

PROPOSITION IX. PROBLEM.

To describe a circle about a given square.

Given the square ABCD; *it is required* to describe a circle about it.

(*Const.*) Join AC and BD, cutting one another in E. (*Dem.*) And because DA is equal to AB, and AC common to the triangles DAC and BAC, the two sides DA and AC are equal to the two BA and AC, and the base DC is equal to the base BC; wherefore the angle DAC is equal to the angle BAC (I. 8), and the angle DAB is bisected by the straight line AC. In the same manner it may be demonstrated that the angles ABC, BCD, and CDA are severally bisected by the straight lines BD and AC; therefore because the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC, the angle EAB is equal to the angle EBA; wherefore the side EA is equal to the side EB (I. 6). In the same manner it may be demonstrated that the straight lines EC and ED are each of them equal to EA or EB; therefore the four straight lines EA, EB, EC, and ED are equal to one another; and the circle described from the centre E, at the distance of one of them, shall pass through the extremities of the other three, and be described about the square ABCD (Def. 6).



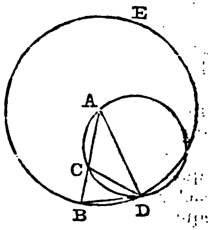
PROPOSITION X. PROBLEM.

To describe an isosceles triangle, having each of the angles at the base double of the third angle.

(*Const.*) Take any straight line AB, and (II. 11) divide it

in the point C, so that the rectangle AB · BC may be equal to the square on CA; from the centre A, at the distance AB, describe the circle BDE, and in it place the straight line BD equal to AC (IV. 1), which is not greater than the diameter of the circle BDE; join DA and DC, and about the triangle ADC describe the circle ACD (IV. 5); the triangle ABD is such as is required; that is, each of the angles ABD and ADB is double of the angle BAD.

(*Dem.*) Because the rectangle AB · BC is equal to the square on AC, and that AC is equal to BD, the rectangle AB · BC is equal to the square on BD; and because from the point B without the circle ACD two straight lines BCA and BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB · BC contained by the secant, and the part of it without the circle, is equal to the square on BD which meets it; the straight line BD touches the circle ACD (III. 37); and because BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal to the angle DAC in the alternate segment of the circle (III. 32); to each of these equals add the angle CDA; therefore the whole angle BDA is equal to the two angles CDA and DAC. But the exterior angle BCD is equal to the angles CDA and DAC (I. 32); therefore also BDA is equal to BCD; but (I. 5) BDA is equal to CBD, because the side AD is equal to the side AB (I. 5); therefore CBD, or DBA, is equal to BCD; and consequently the three angles BDA, DBA, and BCD are equal to one another. And because the angle DBC is equal to the angle BCD, the side BD is equal to the side DC (I. 6). But BD was made equal to CA; therefore also CA is equal to CD, and the angle CDA equal to the angle DAC; therefore the angles CDA and DAC, together, are double of the angle DAC. But BCD is equal to the angles CDA, DAC; therefore also BCD is double of DAC; and BCD is equal to each of the angles BDA and DBA; therefore each of the angles BDA and DBA is double of the angle DAB; wherefore an isosceles triangle ABD is described, having each of the angles at the base double of the third angle.



COR. 1.—Since each of the angles ABD and ADB is double of the angle BAD, the angles of the triangle ABD are together five times the angle BAD, or the angle BAD is one-fifth part of two right angles, and therefore a tenth part of four right angles.

COR. 2.—The arc BD is therefore a tenth part of the whole

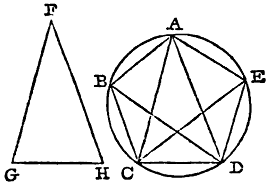
circumference, and hence the chord BD is the side of a decagon inscribed in the circle; but it is the greater segment of the radius divided medially; therefore the side of a regular decagon inscribed in a circle is equal to the greater segment of the radius divided medially.

PROPOSITION XI. PROBLEM.

To inscribe an equilateral and equiangular pentagon in a given circle.

Given the circle ABCDE; *it is required* to inscribe in it an equilateral and equiangular pentagon.

(*Const.*) Describe (IV. 10) an isosceles triangle FGH, having each of the angles at G and H double of the angle at F; and (IV. 2) in the circle ABCDE inscribe the triangle ACD equiangular to the triangle FGH, so that the angle CAD be equal to the angle at F, and each of the angles ACD and CDA equal to the angle at G or H; wherefore each of the angles ACD and CDA is double of the angle CAD. Bisect the angles ACD and CDA (I. 9) by the straight lines CE and DB, and join AB, BC, DE, and EA. ABCDE is the pentagon required.



(*Dem.*) Because the angles ACD and CDA are each of them double of CAD, and they are bisected by the straight lines CE and DB, the five angles ADB, BDC, CAD, DCE, and ECA are equal to one another. But (III. 26) equal angles stand upon equal arcs; therefore the five arcs AB, BC, CD, DE, and EA are equal to one another. And equal arcs are subtended by equal chords (III. 29); therefore the five chords AB, BC, CD, DE, and EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is also equiangular, because the arc AB is equal to the arc DE; if BCD be added to each, the whole ABCD is equal to the whole EDCB; and the angle AED stands on the arc ABCD, and the angle BAE on the arc EDCB; therefore the angle BAE is equal to the angle AED (III. 27). For the same reason, each of the angles ABC, BCD, and CDE is equal to the angle BAE, or AED; therefore the pentagon ABCDE is equiangular; and it has been shewn that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inscribed.

Schol.—That the pentagon is equiangular may be more simply

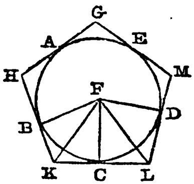
demonstrated by observing that each of the angles stands on the sum of three equal arcs, and they are therefore equal. In the same manner it may be demonstrated that an equilateral polygon of any number of sides, inscribed in a circle, is also equiangular; for each of the angles stands upon the sum of all the arcs except the two, the chords of which contain it.

PROPOSITION XII. PROBLEM.

To describe an equilateral and equiangular pentagon about a given circle.

Given the circle ABCDE; *it is required* to describe about it an equilateral and equiangular pentagon.

(*Const.*) Let the angles of a pentagon, inscribed in the circle, by the last proposition, be in the points A, B, C, D, and E, so that the arcs AB, BC, CD, DE, and EA are equal (IV. 11); and through the points A, B, C, D, and E, draw GH, HK, KL, LM, and MG, touching the circle (III. 17). Take the centre F, and join FB, FK, FC, FL, and FD. (*Dem.*) And because the straight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular to KL (III. 18); therefore each of the angles at C is a right angle. For the same reason, the angles at the points B and D are right angles. And because FCK is a right angle, the square on FK is equal to the squares on FC and CK (I. 47). For the same reason, the square on FK is equal to the squares on FB and BK; therefore the squares on FC and CK are equal to the squares on FB and BK, of which the square on FC is equal to the square on FB; therefore the remaining square on CK is equal to the remaining square on BK, and the straight line CK is equal to BK. And because FB is equal to FC, and FK common to the triangle BFK and CFK, the two BF and FK are equal to the two CF and FK; and the base BK was proved to be equal to the base KC; therefore (I. 8) the angle BFK is equal to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC. For the same reason, the angle CFD is double of the angle CFL, and CLD double of CLF. And because the arc BC is equal to the arc CD, the angle BFC is equal to the angle CFD (III. 27); and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL; therefore in the two triangles FKC, FLC, there are two angles of



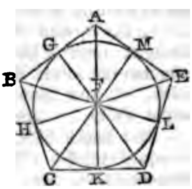
one equal to two angles of the other, each to each; and the side FC, which is adjacent to the equal angles in each, is common to both; therefore the remaining sides and the third angle of the one are equal to the remaining sides and the third angle of the other (I. 26); hence the straight line KC is equal to CL, and the angle FKC to the angle FLC; and because KC is equal to CL, KL is double of KC. In the same manner, it may be shewn that HK is double of BK; and because BK is equal to KC, as was demonstrated, and that KL is double of KC, and HK double of BK, HK is equal to KL. In like manner, it may be shewn that GH, GM, and ML are each of them equal to HK or KL; therefore the pentagon GHKLM is equilateral. It is also equiangular; for, since the angle FKC is equal to the angle FLC, and the angle HKL double of the angle FKC, and KLM double of FLC, as was before demonstrated, the angle HKL is equal to KLM. And in like manner it may be shewn that each of the angles KHG, HGM, and GML is equal to the angle HKL or KLM; therefore the five angles GHK, HKL, KLM, LMG, and MGH being equal to one another, the pentagon GHKLM is equiangular; and it was demonstrated to be equilateral; and it is described about the circle ABCDE.

PROPOSITION XIII. PROBLEM.

To inscribe a circle in a given equilateral and equiangular pentagon.

Given ABCDE an equilateral and equiangular pentagon; *it is required* to inscribe a circle in the pentagon ABCDE.

(*Const.*) Bisect (I. 9) the angles BCD and CDE by the straight lines CF and DF, and from the point F, in which they meet, draw the straight lines FB, FA, FE; and from the point F draw FG, FH, FK, FL, and FM, perpendiculars to AB, BC, CD, DE, and EA.



(*Dem.*) Therefore since BC is equal to CD, and CF common to the triangles BCF and DCF, the two sides BC and CF are equal to the two DC and CF; and the angle BCF is equal to the angle DCF; therefore the base BF is equal to the base FD (I. 4), and the other angles to the other angles, to which the equal sides are opposite; therefore the angle CBF is equal to the angle CDF. And because the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is also double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is bisected by the straight line BF. In

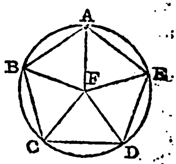
the same manner, it may be demonstrated that the angles BAE and AED are bisected by the straight lines AF and FE. And because the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC and FKC there are two angles of one equal to two angles of the other, and the side FC, which is opposite to one of the equal angles in each, is common to both; therefore the other sides are equal, each to each (I. 26); wherefore the perpendicular FH is equal to the perpendicular FK. In the same manner, it may be demonstrated that FL, FM, and FG are each of them equal to FH or FK; therefore the five straight lines FG, FH, FK, FL, and FM are equal to one another; wherefore the circle described from the centre F, with any of these as a radius, shall pass through the extremities of the other four, and touch the straight lines AB, BC, CD, DE, and EA, because the angles at the points G, H, K, L, and M are right angles; and that a straight line drawn from the extremity of a radius of a circle at right angles to it, touches the circle (III. 16, Cor.); therefore each of the straight lines AB, BC, CD, DE, and EA touches the circle; wherefore it is inscribed in the pentagon ABCDE.

PROPOSITION XIV. PROBLEM.

To describe a circle about a given equilateral and equiangular pentagon.

Given the equilateral and equiangular pentagon ABCDE; *it is required* to describe a circle about it.

(*Const.*) Bisect (I. 9) the angles BCD and CDE by the straight lines CF and FD, and from the point F, in which they meet, draw the straight lines FB, FA, and FE to the points B, A, and E. (*Dem.*) It may be demonstrated, in the same manner as in the preceding proposition, that the angles CBA, BAE, and AED are bisected by the straight lines FB, FA, and FE; and because the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the side CF is equal to the side FD (I. 6). In like manner, it may be demonstrated that FB, FA, and FE are each of them equal to FC or FD; therefore the five straight lines FA, FB, FC, FD, and FE are equal to one another; and the circle described from the centre F, at the distance of one of them, shall pass through the extremities of the other four, and be described about the equilateral and equiangular pentagon ABCDE.

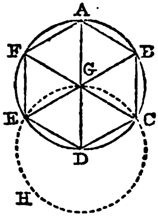


PROPOSITION XV. PROBLEM.

To inscribe an equilateral and equiangular hexagon in a given circle.

Given the circle ABCDEF; *it is required* to inscribe an equilateral and equiangular hexagon in it.

(*Const.*) Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the distance DG, describe the circle EGCH; join EG and CG, and produce them to the points B and F; and join AB, BC, CD, DE, EF, and FA; the hexagon ABCDEF is equilateral and equiangular.



(*Dem.*) Because G is the centre of the circle ABCDEF, GE is equal to GD; and because D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, and DEG are equal to one another (I. 5, Cor.); and the three angles of a triangle are equal to two right angles (I. 32); therefore the angle EGD is the third part of two right angles. In the same manner, it may be demonstrated that the angle DGC is also the third part of two right angles; and because the straight line GC makes with EB the adjacent angles EGC and CGB, equal to two right angles (I. 13); the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, and CGB are equal to one another; and to these are equal the vertical opposite angles BGA, AGF, and FGE (I. 15); therefore the six angles EGD, DGC, CGB, BGA, AGF, and FGE are equal to one another. But equal angles stand upon equal arcs (III. 26); therefore the six arcs AB, BC, CD, DE, EF, and FA are all equal; and equal arcs are subtended by equal chords (III. 29); therefore the six chords are also equal to one another, and the hexagon ABCDEF is equilateral. It is also equiangular, for each of the angles stands on the sum of four equal arcs (Schol. IV. 11). (Or, it is also equiangular; for, since the arc AF is equal to ED, to each of these add the arc ABCD; therefore the whole arc FABCD shall be equal to the whole EDCBA; and the angle FED stands upon the arc FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED. In the same manner, it may be demonstrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED; therefore the hexagon is equiangular); and it was shewn to be equilateral; and it is inscribed in the given circle ABCDEF.

COR.—From this it is manifest, that the side of the hexagon is equal to the radius of the circle.

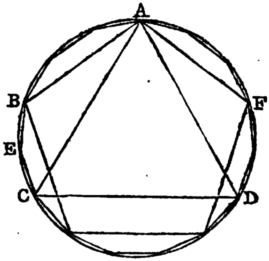
And if through the points A, B, C, D, E, and F there be drawn straight lines touching the circle, an equilateral and equiangular hexagon shall be described about it, which may be demonstrated from what has been said of the pentagon; and likewise a circle may be inscribed in a given equilateral and equiangular hexagon, and circumscribed about it, by a method similar to that used for the pentagon.

PROPOSITION XVI. PROBLEM.

To inscribe an equilateral and equiangular quindecagon in a given circle.

Given the circle ABCD; *it is required* to inscribe an equilateral and equiangular quindecagon in it.

(*Const.*) Let AC be the side of an equilateral triangle inscribed in the circle (IV. 2), and AB the side of an equilateral and equiangular pentagon inscribed in the same (IV. 11); therefore, of such equal parts as the whole circumference ABCDF contains fifteen, the arc ABC, being the third part of the whole, contains five; and the arc AB, which is the fifth part of the whole, contains three; therefore BC, their difference, contains two of the same parts; bisect BC in E (III. 30); (*Dem.*) therefore BE and EC are each of them the fifteenth part of the whole circumference ABCD; therefore if the straight lines BE and EC be drawn, and (IV. 1) straight lines equal to them be placed around in the whole circle, an equilateral and equiangular quindecagon shall be inscribed in it.



COR.—In the same manner, as was done in the pentagon, if through the points of division made by inscribing the quindecagon, straight lines be drawn touching the circle, an equilateral and equiangular quindecagon shall be described about it. And likewise, as in the pentagon, a circle may be inscribed in a given equilateral and equiangular quindecagon, and circumscribed about it.

Schol.—Whatever regular polygon can be inscribed in a circle, another of double the number of sides may be inscribed in it, by bisecting the arc subtended by a side of the former. It has been

proved by M. Gauss, that it is practicable by means of plane geometry—namely, by the intersections of the straight line and circle, or by the resolution of simple and quadratic equations—to inscribe in a circle any regular polygon, provided the number of its sides be a prime number; that is, a number which has no common measure except 1—and be also some power of 2 increased by 1.

He has proved more generally—in his *Disquisitiones Arithmeticæ*—if n, the number of sides of the regular polygon, exceeds by 1 the number which is the product of the prime factors 2^a, 3^b, 5^c, ... that the division of the circle into n equal parts, and consequently the possibility of inscribing in it a regular polygon of n sides, can be reduced to the solution of a equations of the second degree, b of the third, c of the fifth, and so on.

EXERCISES.

1. Through a given point within or without a circle, to draw a chord that shall be equal to a given line.

2. To draw a chord in a circle that shall be equal to a given line, and parallel to another given line, or inclined to it at a given angle.

3. To draw a tangent to a given circle, so that it shall be parallel to a given line.

4. To draw a tangent to a circle, so that it shall make a given angle with a given line.

5. To find a point in a given line that shall be equidistant from another given point in it and from a given line.

6. To describe a circle that shall touch a given line in a given point, and pass through another given point.

7. To draw a line that shall be a common tangent to two circles.

8. To find a point in a given line that shall be equidistant from another given point and a given line.

9. To describe a circle that shall pass through two given points, and touch a given line.

10. To describe a circle that shall touch a given line in a given point, and also touch a given circle.

11. A regular octagon inscribed in a circle is equal to the rectangle under the sides of the inscribed and circumscribing squares.

12. To inscribe a circle in a given quadrant.

13. To describe a circle that shall pass through a given point, and touch a given circle in a given point.

14. If a quadrilateral circumscribe a circle, its opposite sides are together equal to half its perimeter.

15. If every two alternate sides of a polygon be produced to meet, the sum of the salient angles thus formed with eight right angles, will be equal to twice as many right angles as the figure has sides.

16. If on two opposite sides of a rectangle semicircles be described lying on corresponding sides of their diameters, the muxtilineal space contained by their arcs and the other two sides of the rectangle is equal to the rectangle.

17. To describe a circle that shall touch two given lines, and pass through a given point.

18. To describe a circle that shall touch two given lines, and also a given circle.

19. Inscribe a regular hexagon in a given equilateral triangle, and compare its area with that of the triangle.

20. Describe a circle which shall pass through one angle, and touch two sides of a given square.

21. Given a regular pentagon; it is required to describe a triangle having the same area and altitude.

22. In the figure (IV. 10) shew that AC and CD are sides of a pentagon in the less circle; and if DC be produced to meet the circumference in F, and FB be joined, FB will be a side of a pentagon in the greater circle.

23. The square on the side of a pentagon inscribed in a circle, is equal to the sum of the squares on the sides of a hexagon and decagon, inscribed in the same circle.

FIFTH BOOK.

DEFINITIONS.

1. A less magnitude is said to be a *part* of a greater magnitude, when the less *measures* the greater; that is, when the less is contained a certain number of times exactly in the greater. It is also called a *measure*, or *submultiple* of the greater.

2. A line, which is a measure of two or more lines, is called a *common measure* of these lines.

3. Lines that have a common measure, are called *commensurable* lines; those that have no common measure, *incommensurable* lines.

4. A greater magnitude is said to be a *multiple* of a less, when the greater is *measured* by the less; that is, when the greater contains the less a certain number of times exactly.

5. *Equimultiples* of magnitudes are multiples that contain these magnitudes, respectively, the same number of times.

6. *Ratio* is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity; or it is the quotient arising from dividing the first by the second.

7. Magnitudes are said to be *homogeneous*, or of the *same kind*, when the less can be multiplied so as to exeeed the greater; and it is only such magnitudes that are said to have a ratio to one another.

8. The two magnitudes of a ratio are called its *terms*. The first term is called the *antecedent;* the latter, the *consequent.*

9. A ratio is said to be a ratio of *equality*, *majority*, or *minority*, according as the antecedent is equal to, greater, or less than the consequent.

10. If there be four magnitudes, such that if any equimultiples whatsoever be taken of the first and third, and any equimultiples whatsoever of the second and fourth, and if, according as the multiple of the *first* is greater than the multiple of the *second*, equal to it, or less, so is the multiple of the *third* greater than the multiple of the *fourth*, equal to it, or less; then the *first* of the magnitudes is said to have to the *second* the *same ratio* that the *third* has to the *fourth.*

11. Magnitudes are said to be *proportionals* when the first has the same ratio to the second that the third has to the fourth; and the third to the fourth the same ratio which the fifth has to the sixth; and so on, whatever be their number.

When four magnitudes, A, B, C, D, are proportionals, it is usual to say that A is to B as C to D, and to write them thus—A : B :: C : D, or thus, A : B = C : D.

12. When four magnitudes are proportional, they constitute a *proportion* or an *analogy*.

13. The first and last terms of an analogy are called the *extremes;* the second and third, the *means.*

14. When of the equimultiples of four magnitudes, taken as in the tenth definition, the multiple of the first is *greater* than that of the second, but the multiple of the third is *not greater* than the multiple of the fourth; then the first is said to have to the second a *greater ratio* than the third magnitude has to the fourth; and, on the contrary, the third is said to have to the fourth a *less ratio* than the first has to the second.

15. When there is any number of magnitudes greater than two, of which the first has to the second the same ratio that the second has to the third, and the second to the third the same ratio which the third has to the fourth, and so on, the magnitudes are said to be *continual proportionals*, or *in continued proportion.*

16. When three magnitudes are in continued proportion, the second is said to be a *mean proportional* between the other two.

17. When there is any number of magnitudes of the same kind, the *first* is said to have to the *last* of them the *ratio compounded* of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on to the last magnitude. Thus:

If A, B, C, D be four magnitudes of the same kind, the first, A, is said to have to the last, D, the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A : B :: E : F; and B : C :: G : H, and C : D :: K : L; then, since by this definition A has to D the ratio compounded of the ratios of A to B, B to C, C to D; A may also be said to have to D the *ratio compounded* of the ratios of E to F, G to H, and K to L, which are the *same* as these ratios.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D, then, for shortness' sake, M is said to have to N a ratio *compounded of the same ratios* which compound the ratio of A to D; that is, a ratio compounded of the ratios of E to F, G to H, and K to L.

18. A ratio which is compounded of *two equal ratios* is said to be *duplicate* of either of these ratios.

COR.—If the three magnitudes, A, B, and C, are continual proportionals, the ratio of A to C is duplicate of that of A to B, or of B to C; for, by the last definition, the ratio of A to C is compounded of the ratios of A to B, and of B to C; but the ratio of A to B is equal to the ratio of B to C, because A, B, C are continual proportionals; therefore the ratio of A to C, by this definition, is duplicate of the ratio of A to B, or of B to C.

19. A ratio which is compounded of *three equal ratios* is said to be *triplicate* of any one of these ratios; and a ratio which is compounded of *four equal ratios* is said to be *quadruplicate* of any one of these ratios; and so on, according to the number of equal ratios.

COR.—If four magnitudes, A, B, C, D, be continual proportionals, the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D. For the ratio of A to D is compounded of the three ratios of A to B, B to C, C to D; and these three ratios are equal to one another, because A, B, C, D are continual proportionals; therefore the ratio of A to D is triplicate of the ratio of A to B, or of B to C, or of C to D.

20. In proportionals, the antecedent terms are called *homologous* to one another, so also are the consequents.

Geometers make use of the following technical words to signify certain ways of changing either the order or magnitude of the terms of an analogy, so that they continue still to be proportionals:

21. By *alternation*, when the first is to the third, as the second is to the fourth.

22. By *inversion*, when the second is to the first, as the fourth is to the third.

23. By *composition*, when the first, together with the second, is to the second, as the third, together with the fourth, is to the fourth.

24. By *addition*, when the first is to the sum of the first and second, as the third is to the sum of the third and fourth.

25. By *division*, when the excess of the first above the second, is to the second, as the excess of the third above the fourth, is to the fourth.

26. By *conversion*, when the first is to its excess above the second, as the third is to its excess above the fourth.

27. By *mixing*, when the sum of the first and second is to their difference, as the sum of the third and fourth to their difference.

28. By *equality*, when there is any number of magnitudes more than two, and as many others, so that they are proportionals when taken two and two of each rank, and it is inferred—that the *first* is to the *last* of the *first rank* of magnitudes, as the *first* is to the *last* of the *others*:—of this there are the two following kinds, which arise from the different order in which the magnitudes are taken two and two:

29. By *direct equality*, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in a *direct order;* and,

30. By *indirect equality*, when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the last but three to the last but two of the second rank; and so on in an *indirect order.*

The following are some additional algebraical expressions which are convenient for their conciseness:

As mA is A taken m times, it is a multiple of A by m. So $m(A + B)$ is a multiple of $A + B$ by m; $m(A - B)$, a multiple of $A - B$ by m; and $m(A + B - C)$, a multiple of the excess of $A + B$ above C, by m.

Also, mA and mB are equimultiples of A and B by m.

The expression $m + n$ is the sum of the numbers m and n; so mn is the product of these numbers. Also, mnA is a multiple of A by a number which is the product of m and n; and $(m + n)A$, is a multiple of A by a number which is the sum of m and n.

AXIOMS.

1. Equimultiples of the same, or of equal magnitudes, are equal to one another.

2. Those magnitudes of which the same, or equal magnitudes, are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

PROPOSITION I. THEOREM.

If any number of magnitudes be equimultiples of as many others, each of each, what multiple soever any one of the first is

of its part, the same multiple is the sum of all the first of the sum of all the rest.

Given any number of magnitudes, A, B, and C, equimultiples of as many others D, E, and F, each of each; *to prove that* $A + B + C$ is the same multiple of $D + E + F$ that A is of D.

(*Prep.*) Let A contain D, B contain E, and C contain F, each any number of times, as, for instance, three times. Then, because A contains D three times,

$$A = D + D + D.$$

For the same reason, $B = E + E + E$;

And also, $C = F + F + F$.

(*Dem.*) Therefore, adding equals to equals, $A + B + C$ is equal to $D + E + F$ taken three times. In the same manner, if A, B, and C were each any other equimultiple of D, E, and F, it would be shewn that $A + B + C$ was the same multiple of $D + E + F$.

Cor.—Hence, if m be any number, $mD + mE + mF = m(D + E + F)$. For mD, mE, and mF are multiples of D, E, and F, by m; therefore their sum is also a multiple of $D + E + F$, by m.

PROPOSITION II. THEOREM.

If to a multiple of a magnitude by any number a multiple of the same magnitude by any number be added, the sum will be the same multiple of that magnitude that the sum of the two numbers is of unity.

Given $A = mC$, and $B = nC$; *to prove that* $A + B = (m + n)C$.

(*Dem.*) For, since $A = mC$, $A = C + C + C + \&c.$, C being repeated m times. For the same reason, $B = C + C + \&c.$, C being repeated n times. Therefore, adding equals to equals, $A + B$ is equal to C taken $m + n$ times; that is, $A + B = (m + n)C$. Therefore $A + B$ contains C as oft as there are units in $m + n$.

Cor. 1.—In the same way, if there be any number of multiples whatsoever, as $A = mE$, $B = nE$, $C = pE$, it is shewn that $A + B + C = (m + n + p)E$.

Cor. 2.—Hence also, since $A + B + C = (m + n + p)E$, and since $A = mE$, $B = nE$, and $C = pE$, $mE + nE + pE = (m + n + p)E$.

PROPOSITION III. THEOREM.

If the first of three magnitudes contain the second as oft as there are units in a certain number, and if the second contain the third also as often as there are units in a certain number, the first will contain the third as oft as there are units in the product of these two numbers.

Given $A = mB$, and $B = nC$; *to prove that* $A = mnC$.

(*Dem.*) Since $B = nC$, $mB = nC + nC + \&c.$, repeated m times. But $nC + nC + \&c.$, repeated m times, is (V. 2) equal to C multiplied by $n + n + \&c.$, n being repeated m times. And n repeated m times is mn; therefore $mB = mnC$. But by hypothesis, $A = mB$, therefore $A = mnC$.

PROPOSITION IV. THEOREM.

If any equimultiples be taken of the antecedents of an analogy, and any equimultiples of the consequents, these multiples, taken in the order of the terms, are proportional.

Given that $A : B :: C : D$, and let m and n be any two numbers; *to prove that* $mA : nB :: mC : nD$.

(*Prep.*) Take of mA and mC equimultiples by any number p, and of nB and nD equimultiples by any number q. (*Dem.*) Then the equimultiples of mA and mC by p are equimultiples also of A and C, for they contain A and C as oft as there are units in pm (V. 3), and are equal to pmA and pmC. For the same reason, the multiples of nB and nD by q are qnB, qnD. Since therefore $A : B :: C : D$, and of A and C there are taken any equimultiples—namely, pmA and pmC, and of B and D, any equimultiples qnB, qnD, if pmA be greater than qnB, pmC must be greater than qnD (V. Def. 10); if equal, equal; and if less, less. But pmA, pmC are also equimultiples of mA and mC by p, and qnB, qnD are equimultiples of nB and nD by q; therefore $mA : nB :: mC : nD$.

COR.—In the same manner, it may be demonstrated that if $A : B :: C : D$, and of A and C equimultiples be taken by any number m—namely, mA and mC, $mA : B :: mC : D$. This may also be considered as included in the proposition, and as being the case when $n = 1$.

PROPOSITION V. THEOREM.

If one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other; the difference of the two multiples shall be the same multiple of the difference of the two quantities that the whole is of the whole.

Given mA and mB any equimultiples of the two magnitudes A and B, of which A is greater than B; *to prove that* $mA - mB$ is the same multiple of $A - B$ that mA is of A; that is, $mA - mB = m(A - B)$.

(*Prep.*) Let D be the excess of A above B, then $A - B = D$, and adding B to both $A = D + B$. (*Dem.*) Therefore (V. 1) $mA = mD + mB$; take mB from both, and $mA - mB = mD$; but $D = A - B$, therefore $mA - mB = m(A - B)$.

PROPOSITION VI. THEOREM.

If from a multiple of a magnitude by any number a multiple of the same magnitude by a less number be taken away, the remainder will be the same multiple of that magnitude that the difference of the numbers is of unity.

Given mA and nA multiples of the magnitude A by the numbers m and n, and let m be greater than n; *to prove that* $mA - nA$ contains A as oft as $m - n$ contains unity, or $mA - nA = (m - n)A$.

(*Dem.*) Let $m - n = q$; then $m = n + q$. Therefore (V. 2) $mA = nA + qA$; take nA from both, and $mA - nA = qA$. Therefore $mA - nA$ contains A as oft as there are units in q; that is, in $m - n$, or $mA - nA = (m - n)A$.

COR.—When the difference of the two numbers is equal to unity, or $m - n = 1$, then $mA - nA = A$.

PROPOSITION A. THEOREM.

If four magnitudes be proportionals, they are proportionals also when taken inversely.

Given $A : B :: C : D$; *to prove that* $B : A :: D : C$.

(*Prep.*) Let mA and mC be any equimultiples of A and C; nB and nD any equimultiples of B and D. (*Dem.*) Then, because $A : B :: C : D$, if mA be less than nB, mC will be less than nD (V. Def. 10); that is, if nB be greater than mA, nD will be greater than mC. For the same reason, if $nB = mA$, $nD = mC$, and if $nB \angle mA$, $nD \angle mC$. But nB, nD are any equimultiples of B and D, and mA, mC any equimultiples of A and C; therefore (V. Def. 10) $B : A :: D : C$.

PROPOSITION B. THEOREM.

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third to the fourth.

Given mA, mB equimultiples of the magnitudes A and B; *to prove that* $mA : A :: mB : B$, and *that* $A : mA :: B : mB$.

(*Prep.*) Take of mA and mB equimultiples by any number n; and of A and B equimultiples by any number p; these will be nmA (V. 3), pA, nmB, pB. (*Dem.*) Now, if nmA be greater than pA, nm is also greater than p; and if nm is greater than p, nmB is greater than pB; therefore, when nmA is greater than pA, nmB is greater than pB. In the same manner, if $nmA = pA$, $nmB = pB$, and if $nmA \angle pA$, $nmB \angle pB$. Now, nmA, nmB are any equimultiples of mA and mB; and pA, pB are any equimultiples of A and B; therefore $mA : A :: mB : B$ (V. Def. 10).

Again, since $mA : A :: mB : B$, by inversion (V. A), $A : mA :: B : mB$.

PROPOSITION C. THEOREM.

If the first term of an analogy be a multiple or a part of the second, the third is the same multiple or the same part of the fourth.

Given $A : B :: C : D$, and first let A be a multiple of B; *to prove that* C is the same multiple of D; that is, if $A = mB$, $C = mD$.

(*Prep.*) Take of A and C equimultiples by any number as 2—namely, 2A and 2C; and of B and D, take equimultiples by the number $2m$—namely, $2mB$, $2mD$ (V. 3); (*Dem.*) then, because $A = mB$, $2A = 2mB$; and since $A : B :: C : D$, and since $2A = 2mB$, therefore $2C = 2mD$ (V. Def. 10), and $C = mD$; that is, C contains D m times, or as often as A contains B.

Next, let A be a part of B, C is the same part of D. For, since $A : B :: C : D$, inversely (V. A), $B : A :: D : C$. But A being a part of B, B is a multiple of A, and therefore, as is shewn above, D is the same multiple of C, and therefore C is the same part of D that A is of B.

PROPOSITION VII. THEOREM.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Given A and B any equal magnitudes, and C any other; *to prove that* $A : C :: B : C$.

(*Prep.*) Let mA, mB be any equimultiples of A and B; and nC any multiple of C.

(*Dem.*) Because $A = B$, $mA = mB$ (V. Ax. 1); wherefore,

if mA be greater than nC, mB is greater than nC; and if mA $= n$C, mB $= n$C; or, if mA $\angle n$C, mB $\angle n$C. But mA and mB are any equimultiples of A and B, and nC is any multiple of C; therefore (V. Def. 10) A : C :: B : C.

Again, if A $=$ B, C : A :: C : B; for, as has been proved, A : C :: B : C, and inversely, (V. A), C : A :: C : B.

PROPOSITION VIII. PROBLEM.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Given A $+$ B a magnitude greater than A, and C a third magnitude; *to prove that* A $+$ B has to C a greater ratio than A has to C; and C has a greater ratio to A than it has to A $+$ B.

(*Prep.*) Let m be such a number that mA and mB are each of them greater than C; and let nC be the least multiple of C that exceeds mA $+ m$B; then nC $-$ C; that is (V. 1), $(n-1)$C will be less than mA $+ m$B; or mA $+ m$B, that is, m(A $+$ B) is greater than $(n-1)$C. (*Dem.*) But because nC is greater than mA $+ m$B, and C less than mB, nC $-$ C is greater than mA, or mA is less than nC $-$ C; that is, than $(n-1)$C. Therefore the multiple of A $+$ B by m exceeds the multiple of C by $n-1$, but the multiple of A by m does not exceed the multiple of C by $n-1$; therefore A $+$ B has a greater ratio to C than A has to C (V. Def. 14).

Again, because the multiple of C by $n-1$, exceeds the multiple of A by m, but does not exceed the multiple of A $+$ B by m, C has a greater ratio to A than it has to A $+$ B.

PROPOSITION IX. THEOREM.

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

Given A : C :: B : C; *to prove that* A $=$ B.

(*Prep.*) For, if not, let A be greater than B; then, because A is greater than B, two numbers, m and n, may be found as in the last proposition, such that mA shall exceed nC, while mB does not exceed nC. (*Dem.*) But because A : C :: B : C; if mA exceed nC, mB must also exceed nC (V. Def. 10); and it is also shewn that mB does not exceed nC, which is impossible. Therefore A is not greater than B; and in the same way it is demonstrated that B is not greater than A; therefore A is equal to B.

Next, let $C : A :: C : B$, $A = B$. For by inversion (V. A), $A : C :: B : C$; and therefore by the first case, $A = B$.

PROPOSITION X. THEOREM.

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

Given the ratio of A to C greater than that of B to C; *to prove that* A is greater than B.

(*Dem.*) Because $A : C > B : C$, two numbers m and n may be found, such that $mA > nC$, and $mB < nC$ (V. Def. 14). Therefore, also $mA > mB$, and $A > B$ (V. Ax. 4).

Again, let $C : B > C : A$; $B < A$. For two numbers, m and n may be found, such that $nC > mB$, and $nC < mA$. Therefore, since mB is less, and mA greater than the same magnitude, nC, $mB < mA$, and therefore $B < A$.

PROPOSITION XI. THEOREM.

Ratios that are equal to the same ratio are equal to one another.

Given $A : B :: C : D$; and also $C : D :: E : F$; *to prove that* $A : B :: E : F$.

(*Prep.*) Take mA, mC, mE any equimultiples of A, C, and E; and nB, nD, nF any equimultiples of B, D, and F. (*Dem.*) Because $A : B :: C : D$, if $mA > nB$, $mC > nD$ (V. Def. 10); but if $mC > nD$, $mE > nF$, because $C : D :: E : F$; therefore if $mA > nB$, $mE > nF$. In the same manner, if $mA = nB$, $mE = nF$; and if $mA < nB$, $mE < nF$. Now, mA, mE are any equimultiples whatever of A and E; and nB, nF any whatever of B and F; therefore $A : B :: E : F$ (V. Def. 10).

PROPOSITION XII. THEOREM.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so is the sum of all the antecedents to that of the consequents.

Given $A : B :: C : D$, and $C : D :: E : F$; *to prove that* $A : B :: A + C + E : B + D + F$.

(*Prep.*) Take mA, mC, mE any equimultiples of A, C, and E; and nB, nD, nF any equimultiples of B, D, and F. (*Dem.*) Then, because $A : B :: C : D$, if $mA > nB$, $mC > nD$ (V. *Def.* 8); and when $mC > nD$, $mE > nF$, because $C : D :: E : F$.

Therefore, if $mA > nB$, $mA + mC + mE > nB + nD + nF$. In the same manner, if $mA = nB$, $mA + mC + mE = nB + nD + nF$; and if $mA < nB$, $mA + mC + mE < nB + nD + nF$. Now, $mA + mC + mE = m(A + C + E)$ (V. 1), so that mA and $mA + mC + mE$ are any equal multiples of A and of $A + C + E$. And for the same reason nB, and $nB + nD + nF$, are any equimultiples of B, and of $B + D + F$; therefore (V. Def. 10) $A : B :: A + C + E : B + D + F$.

PROPOSITION XIII. THEOREM.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth has to the sixth.

Given $A : B :: C : D$; but $C : D > E : F$; *to prove that* $A : B > E : F$.

(*Dem.*) Because $C : D > E : F$, there are some two numbers m and n, such that $mC > nD$, but $mE < nF$ (V. Def. 14). Now, if $mC > nD$, $mA > nB$, because $A : B :: C : D$. Therefore $mA > nB$, and $mE < nF$; wherefore $A : B > E : F$.

PROPOSITION XIV. THEOREM.

If the first term of an analogy be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Given $A : B :: C : D$; *to prove that* if $A > C$, $B > D$, if $A = C$, $B = D$, and if $A < C$, $B < D$.

(*Dem.*) First, let $A > C$; then $A : B > C : B$ (V. 8), but $A : B :: C : D$, therefore $C : D > C : B$ (V. 13), and therefore $B > D$ (V. 10).

In the same manner, it is proved that if $A = C$, $B = D$; and if $A < C$, $B < D$.

PROPOSITION XV. THEOREM.

Magnitudes have the same ratio to one another which their equimultiples have.

Given A and B two magnitudes, and m any number; *to prove that* $A : B :: mA : mB$.

(*Dem.*) Because $A : B :: A : B$ (V. 7); $A : B :: A + A : B + B$ (V. 12), or $A : B :: 2A : 2B$. And since it has been proved that $A : B :: 2A : 2B$, $A : B :: A + 2A :: B + 2B$ (V. 12), or $A : B :: 3A : 3B$; and so on, for all the equimultiples of A and B.

PROPOSITION XVI. THEOREM.

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Given $A : B :: C : D$; *to prove that* alternately, $A : C :: B : D$.

(*Prep.*) Take mA, mB any equimultiples of A and B, and nC, nD any equimultiples of C and D. (*Dem.*) Then (V. 15) $A : B :: mA : mB$; now $A : B :: C : D$; therefore (V. 11) $C : D :: mA : mB$. But $C : D :: nC : nD$ (V. 15); therefore $mA : mB :: nC : nD$ (V. 11); wherefore if $mA > nC$, $mB > nD$ (V. 14); if $mA = nC$, $mB = nD$, or if $mA < nC$, $mB < nD$; therefore (V. Def. 10) $A : C :: B : D$.

PROPOSITION XVII. THEOREM.

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth, the first shall have to the second the same ratio which the third has to the fourth.

Given $A + B : B :: C + D : D$; *to prove that* by division, $A : B :: C : D$.

(*Prep.*) Take mA and nB any multiples of A and B, by the numbers m and n; and first, let $mA > nB$; to each of them add mB, then $mA + mB > mB + nB$. But $mA + mB = m(A + B)$ (V. 1), and $mB + nB = (m + n)B$ (V. 2); therefore $m(A + B) > (m + n)B$.

(*Dem.*) And because $A + B : B :: C + D : D$, if $m(A + B) > (m + n).B$, $m(C + D) > (m + n)D$, or $mC + mD > mD + nD$; that is, taking mD from both, $mC > nD$. Therefore, when mA is greater than nB, mC is greater than nD. In like manner it is demonstrated that if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$; therefore $A : B :: C : D$ (V. Def. 10).

PROPOSITION XVIII. THEOREM.

The terms of an analogy are proportional by composition.

Given $A : B :: C : D$; *to prove that* by composition, $A + B : B :: C + D : D$.

(*Prep.*) Take $m(A + B)$ and nB any multiples whatever of $A + B$ and B; and first, let m be greater than n. (*Dem.*) Then, because $A + B$ is also greater than B, $m(A + B) > n.B$.

For the same reason, $m(C + D) > nD$. In this case, therefore, that is, when $m > n$, $m(A + B)$ is greater than nB, and $m(C + D)$ is greater than nD. And in the same manner it may be proved that when $m = n$, $m(A + B)$ is greater than nB, and $m(C + D)$ greater than nD.

Next, let $m < n$, or $n > m$, then $m(A + B)$ may be greater than nB, or may be equal to it, or may be less; first, let $m(A + B)$ be greater than nB; then also, $mA + mB > nB$; take mB, which is less than nB, from both, and $mA > nB - mB$, or $mA > (n - m)B$ (V. 6). But if $mA > (n - m)B$, $mC > (n - m)D$; because $A : B :: C : D$. Now, $(n - m)D = nD - mD$, therefore $mC > nD - mD$, and adding mD to both $mC + mD > nD$, that is (V. 1), $m(C + D) > nD$. If therefore $m(A + B) > nB$, $m(C + D) > nD$.

In the same manner it will be proved that if $m(A + B) = nB$, $m(C + D) = nD$; and if $m(A + B) < nB$, $m(C + D) < nD$; therefore $A + B : B :: C + D : D$.

PROPOSITION XIX. THEOREM.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the remainder, as the whole to the whole.

Given $A : B :: C : D$, and C less than A; *to prove that* $A - C : B - D :: A : B$.

(*Dem.*) Because $A : B :: C : D$, alternately (V. 16), $A : C :: B : D$; and therefore by division (V. 17), $A - C : C :: B - D : D$. Wherefore, again, alternately, $A - C : B - D :: C : D$; but $A : B :: C : D$, therefore (V. 11) $A - C : B - D :: A : B$.

COR. $A - C : B - D :: C : D$.

PROPOSITION D. THEOREM.

The terms of an analogy are proportional by conversion.

Given $A : B :: C : D$; *to prove that* by conversion,

$$A : A - B :: C : C - D.$$

(*Dem.*) For since $A : B :: C : D$, by division (V. 17), $A - B : B :: C - D : D$, and inversely (V. A), $B : A - B :: D : C - D$; therefore, by composition (V. 18),

$$A : A - B :: C : C - D.$$

PROPOSITION XX. THEOREM.

If there be three magnitudes, and other three, which, taken two and two in order, have the same ratio; if the first be

greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Given three magnitudes, A, B, and C, and other three D, E, and F; such that $A:B::D:E$; and $B:C::E:F$; *to prove that* if $A > C$, $D > F$; if $A = C$, $D = F$; and if $A < C$, $D < F$.

A, B, C,
D, E, F.

(*Dem.*) First, let $A > C$; then $A:B > C:B$ (V. 8). But $A:B::D:E$, therefore also $D:E > C:B$ (V. 13). Now $B:C::E:F$, and inversely (V. A), $C:B::F:E$; and it has been shewn that $D:E > C:B$, therefore $D:E > F:E$; and consequently $D > F$ (V. 10).

Next, let $A = C$; then $A:B::C:B$ (V. 7), but $A:B::D:E$, therefore, $C:B::D:E$; but $C:B::F:E$, therefore, $D:E::F:E$ (V. 11), hence $D = F$ (V. 9). Lastly, let $A < C$. Then $C > A$; and, as was shewn in the first case, $C:B::F:E$, and $B:A::E:D$; therefore, by the first case, if $C > A$, $F > D$; that is, if $A < C$, $D < F$.

PROPOSITION XXI. THEOREM.

If there be three magnitudes, and other three, which have the same ratio taken two and two, but in a cross order; if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Given three magnitudes, A, B, C, and other three, D, E, and F, such that $A:B::E:F$, and $B:C::D:E$; *to prove that* if $A > C$, $D > F$; if $A = C$, $D = F$, and if $A < C$, $D < F$.

(*Dem.*) First, let $A > C$. Then $A:B > C:B$ (V. 8); but $A:B::E:F$, therefore, $E:F > C:B$ (V. 13). Now, $B:C::D:E$, and inversely, $C:B::E:D$; therefore, $E:F > E:D$; wherefore, $D > F$ (V. 10).

A, B, C,
D, E, F.

Next, let $A = C$. Then (V. 7) $A:B::C:B$; but $A:B::E:F$; therefore, $C:B::E:F$ (V. 11); but $B:C::D:E$, and inversely, $C:B::E:D$; therefore, $E:F::E:D$, and consequently, $D = F$ (V. 9).

Lastly, let $A < C$. Then $A:B < C:B$, but $A:B::E:F$, therefore, $E:F < C:B$, but $B:C::D:E$, and inversely (V. A), $C:B::E:D$, therefore, $E:F < E:D$, and therefore $D < F$ (V. 10).

PROPOSITION XXII. THEOREM.

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be *given* three magnitudes, A, B, and C, and other three, D, E, and F, which, taken two and two in order, have the same ratio; namely, A : B :: D : E, and B : C :: E : F; *to prove that* A : C :: D : F.

(*Prep.*) Take of A and D any equimultiples whatever, mA, mD; of B and E any whatever, nB, nE; and of C and F any whatever, qC, qF.

A,	B,	C,
D,	E,	F,
mA,	nB,	qC,
mD,	nE,	qF.

(*Dem.*) Because A : B :: D : E, $mA : nB :: mD : nE$ (V. 4); and for the same reason, $nB : qC :: nE : qF$. Therefore (V. 20), according as mA is greater than qC, equal to it, or less, mD is greater than qF, equal to it, or less; but mA, mD are any equimultiples of A and D; and qC, qF are any equimultiples of C and F; therefore (V. Def. 10) A : C :: D : F.

Again, let there be *given* four magnitudes, and other four, which, taken two and two, have the same ratio; namely, A : B :: E : F; B : C :: F : G; C : D :: G : H; *to prove that* A : D :: E : H.

(*Dem.*) For since A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the same ratio, by the foregoing case A : C :: E : G. And because also C : D :: G : H, by that same case, A : D :: E : H.

A,	B,	C,	D,
E,	F,	G,	H.

In the same manner the demonstration is extended to any number of magnitudes.

PROPOSITION XXIII. THEOREM.

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be *given* three magnitudes, A, B, C, and other three, D, E, and F, which, taken two and two in a cross order, have the same ratio; namely, A : B :: E : F, and B : C :: D : E; *to prove that* A : C :: D : F.

(*Prep.*) Take of A, B, and D any equimultiples mA, mB, mD; and of C, E, F any equimultiples nC, nE, nF.

(*Dem.*) Because $A : B :: E : F$, and because also $A : B :: mA : mB$ (V. 15), and $E : F :: nE : nF$; therefore, $mA : mB :: nE : nF$ (V. 11). Again, because $B : C :: D : E$, $mB : nC :: mD : nE$ (V. 4); and it has been just shewn that $mA : mB :: nE : nF$; therefore, if $mA > nC$, $mD > nF$ (V. 21); if $mA = nC$, $mD = nF$; and if $mA < nC$, $mD < nF$. Now, mA and mD are any equimultiples of A and D, and nC, nF any equimultiples of C and F; therefore, $A : C :: D : F$ (V. Def. 10).

A,	B,	C,
D,	E,	F.
mA,	mB,	nC,
mD,	nE,	nF.

Next, let there be *given* four magnitudes, A, B, C, and D, and other four, E, F, G, and H, which, taken two and two in a cross order, have the same ratio; namely, $A : B :: G : H$, $B : C :: F : G$, and $C : D :: E : F$, then $A : D :: E : H$.

A,	B,	C,	D,
E,	F,	G,	H.

For, since A, B, C are three magnitudes, and F, G, H other three, which, taken two and two in a cross order, have the same ratio, by the first case $A : C :: F : H$. But $C : D :: E : F$; therefore A, C, and D are three magnitudes, and E, F, and H other three, which, taken two and two in a cross order, are proportional, hence, by the first case, $A : D :: E : H$. In the same manner the demonstration may be extended to any number of magnitudes.

PROPOSITION XXIV. THEOREM.

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Given $A : B :: C : D$, and also $E : B :: F : D$, *to prove that* $A + E : B :: C + F : D$.

Because $E : B :: F : D$, by inversion, $B : E :: D : F$. But by hypothesis, $A : B :: C : D$; therefore, by equality (V. 22), $A : E :: C : F$, and by composition (V. 18), $A + E : E :: C + F : F$. Now, again by hypothesis, $E : B :: F : D$; therefore, by equality, $A + E : B :: C + F : D$.

PROPOSITION E. THEOREM.

Ratios which are compounded of the same ratios are the same with one another.

Given the ratios of A to B, and of B to C, which compound the ratio of A to C, equal, each to each, to the ratios of D to E, and E to F, which compound the ratio of D to F; *to prove that* $A : C :: D : F$.

(*Dem.*) For, first, if the ratio of A to B be equal to that of D to E, and the ratio of B to C equal to that of E to F, by equality (V. 22), $A : C :: D : F$.

A,	B,	C,
D,	E,	F.

And next, if the ratio of A to B be equal to that of E to F, and the ratio of B to C equal to that of D to E, by indirect equality (V. 23), $A : C :: D : F$. In the same manner may the proposition be demonstrated whatever be the number of ratios.

PROPOSITION F.

The terms of an analogy are proportional by addition.

Given $A : B :: C : D$; *to prove that* by addition, $A : A + B :: C : C + D$.

(*Dem.*) For $B : A :: D : C$, by inversion (V. A); therefore $A + B : A :: C + D : C$, by composition (V. 18); hence, by inversion, $A : A + B = C : C + D$ (V. A).

PROPOSITION G.

The terms of an analogy are proportional by mixing.

Given $A : B :: C : D$; *to prove that* $A + B : A - B :: C + D : C - D$.

(*Dem.*) For $A + B : B = C + D : D$ by composition (V. 18); and by division, $A - B : B :: C - D : D$ (V. 17), also by inversion (V. A).

$$B : A - B :: D : C - D;$$

but

$$A + B : B :: C + D : D;$$

therefore, by equality (V. 22), $A + B : A - B :: C + D : C - D$.

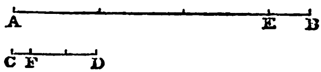
If $B > A$, it may be similarly proved that $A + B : B - A :: C + D : D - C$.

PROPOSITION H. PROBLEM.

To find a common measure of two lines.

Given the two lines AB and CD, *it is required* to find a common measure of them.

Find the number of times that CD is contained in AB. If it be contained an exact number of times, then it is a measure of it, and any part of it will also be a common measure. But if it be contained several times in AB, as three times, with a remainder EB; then if EB be a measure of CD, it will also be one of AE, which is a multiple of CD; and therefore it will be also a measure of AB; it would therefore be the common measure required. But if EB be not a measure of CD, let it be contained in it a certain number of times, as 2 times, with the remainder CF; then if CF be a measure of EB, it will also be a measure of DF, which is the multiple of EB; and therefore it will also be a measure of CD, and consequently of AE, and therefore also of AB. Let CF be contained 2 times in EB, then $EB = 2CF$; $CD = 2EB + CF = 4CF + CF = 5CF$; and $AB = 3CD + EB = 15CF + 2CF = 17CF$. And CF is therefore contained 5 times in CD, and 17 times in AB.



In the same manner the common measure of any other two commensurable lines may be found.

Cor. 1.—If the process for finding a common measure of two lines never terminates, the lines are incommensurable.

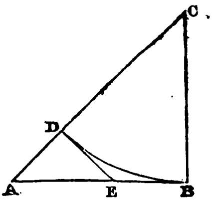
Cor. 2.—Any part of a common measure is also a common measure; and the measure found as above is the greatest common measure.

Cor. 3.—Any two commensurable lines are to one another, as the numbers denoting the number of times that they respectively contain their common measure.

PROPOSITION K. THEOREM.

The diagonal and side of a square are incommensurable.

Given ABC the half of a square; *to prove that* the side AB and diagonal AC are incommensurable.



(*Dem.*) Since the angle at B is a right angle, each of the equal angles at C and A is less than a right angle (I. 32); therefore $AB \angle AC$. Again, $AB + BC > AC$ (I. 20), or $2AB > AC$; hence AC is greater than once AB, and less than twice AB; and the same may be proved of the diagonal and side of any square. Therefore, when the side of a square is taken once from its diagonal, there is always a remainder less than its side.

From C as a centre, with the radius CB, describe the arc DB; then CD = CB = AB, and AD $\angle$ AB. From D draw DE perpendicular to AC; then ED and EB being tangents (III. 16, Cor.), ED = EB (III. 37, Cor.). But in the triangle ADE, the angle at D is a right angle, and the angle at A is half a right angle; therefore the angle at E is half a right angle, and therefore AD = DE = EB. When the first remainder AD therefore is taken from AB, the remaining part AE, from which AD is still to be taken, is the diagonal of a square, of which AD is the side. But this is the same as the former process; and when it has been performed in regard to AE and AD, the remaining lines to be compared will therefore again be the side and diagonal of a still smaller square; but since, when the side of a square is taken from its diagonal, there is a remainder, therefore in the above process there will always be found to be a remainder; the process therefore will never terminate, or no common measure can ever be found; therefore AC and AB are incommensurable.

EXERCISES.

1. If all the terms, or any two homologous terms, or the terms of either of the ratios, of an analogy, be multiplied or divided by the same number, the resulting magnitudes are still proportional.

2. If any number of magnitudes be in continued proportion, the difference between the first and second terms is to the first, as the difference between the first and last to the sum of all the terms except the last.

3. If the same magnitude be added to the terms of a ratio, it will be unchanged, increased, or diminished, according as it is a ratio of equality, minority, or majority.

4. The differences of the successive terms of continued proportionals are also in continued proportion.

5. The first term of an infinite decreasing series of quantities in continued proportion, is a mean proportional between its excess above the second, and the sum of the series.

FIFTH BOOK.

SUPPLEMENT.

Since the principles of proportion, as delivered in the Fifth Book of Euclid are generally found very difficult, and many teachers prefer to have them established in a more simple way; the following Fifth Book may be substituted for it, in which, it is hoped, the propositions are all proved in as simple a manner as the subject will admit of.

In order to effect this, it will be necessary to adopt a different definition of proportion from that given by Euclid—namely, '*If four quantities be proportional, the first divided by the second gives the same quotient as the third divided by the fourth.*' But as there is no geometrical method of dividing one line by another, proportion cannot be applied to Geometry by this definition; it will therefore be necessary to prove that if quantities are proportional by the one definition, they are also proportional by the other, and conversely; and then apply the results so obtained by Euclid's definition, which is the *tenth* of the previous Fifth Book.

The definitions of the Fifth Book are also to be used as definitions in this Supplementary Book.

ARITHMETICAL PRINCIPLES.

In order to establish the doctrines of Proportion in the manner indicated above, it will be necessary to apply the following arithmetical principles.

1*st*, A fraction is multiplied by a whole number by multiplying its numerator; thus $\frac{5}{12}$ is multiplied by 7, by multiplying its numerator 5 by 7 and retaining the same denominator, thus $\frac{5 \times 7}{12} = \frac{35}{12}$; for the result $\frac{35}{12}$ contains 7 times as many of the same kind of parts as the multiplicand, it is therefore multiplied by 7.

2*d*, A fraction is divided by a whole number by multiplying its denominator by that number and retaining the same numerator; thus $\frac{5}{3}$ is divided by 6 by multiplying its denominator by

6 and retaining the same numerator; thus $\frac{5}{3 \times 6} = \frac{5}{18}$; for an eighteenth part is only the sixth part of a third part, and hence $\frac{5}{18}$ is the sixth part of $\frac{5}{3}$, the fraction has therefore been divided by 6 by multiplying its denominator by 6.

3d, A fraction is multiplied by a fraction by multiplying the numerators together for the numerator of the product, and the denominators together for the denominator of the product; thus, to multiply $\frac{3}{4}$ by $\frac{5}{8}$, $3 \times 5 = 15$ the numerator of the product, and $4 \times 8 = 32$ the denominator of the product; or, more concisely, $\frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$; this is evident for $\frac{5}{8}$ is 5 times $\frac{1}{8}$, hence it is the same as to multiply by 5 and then divide by 8, which (by 1 and 2) gives the above result.

4th, A fraction is divided by a fraction by inverting the divisor, and then multiplying the numerators for the numerator of the quotient and the denominators for the denominator; thus, to divide $\frac{3}{7}$ by $\frac{4}{5}$, invert the $\frac{4}{5}$ by making it $\frac{5}{4}$, then $\frac{3}{7} \times \frac{5}{4} = \frac{15}{28}$. Now $\frac{4}{5}$ is 4 times $\frac{1}{5}$, and to divide the fraction $\frac{3}{7}$ by 4 is to multiply its denominator by 4, which gives $\frac{3}{28}$ (2), but it was only the $\frac{1}{5}$ part of 4 that was the divisor; hence, as it has been divided by 5 times the divisor, the quotient is therefore only the fifth part of the true quotient, therefore the numerator must be multiplied by 5 to obtain the true quotient (1), wherefore $\frac{3}{7} \times \frac{5}{4} = \frac{15}{28}$ is the true quotient.

5th, A fraction is not altered in value by multiplying or dividing both its numerator and denominator by the same number. For $\frac{2}{3} \times \frac{12}{12} = \frac{24}{36}$, if these be each parts of a yard, the first $\frac{2}{3}$ evidently represents 2 feet; but the second $\frac{24}{36}$ will represent 24 inches, which is equal to 2 feet, hence its value is not altered. Or, since multiplying the numerator by 12, multiplies (1) the fraction by 12; and multiplying the denominator by 12, divides (2) the fraction by 12; it has therefore been multiplied and divided by the same number, hence its value is not altered. In the same manner, if $\frac{24}{36}$ be divided by 12, both numerator and denominator, the result is $\frac{2}{3}$, a fraction of equal value.

J

AXIOMS.

1. If equals be multiplied by the same, or by equals, the products are equals.

2. If equals be divided by the same, or by equals, the quotients are equals.

3. A fraction is greater than, equal to, or less than unity, according as its numerator is greater than, equal to, or less than its denominator.

4. According as the numerator of a fraction is greater than, equal to, or less than its denominator, so is the value of the fraction greater than, equal to, or less than unity.

5. The greater of two unequal quantities divided by any quantity, gives a greater quotient than the less when divided by the same quantity.

6. If two quantities when divided by the same or equal quantities give unequal quotients, that quantity which gives the greater quotient is the greater of the two.

PROPOSITION I. THEOREM.

If four magnitudes be such that the first divided by the second gives the same quotient as the third divided by the fourth; then if any equimultiples be taken of the first and third, and any equimultiples of the second and fourth, according as the multiple of the first is greater than that of the second, equal to it, or less, so is the multiple of the third greater than that of the fourth, equal to it, or less.

Given A, B, C, and D four magnitudes such that $\frac{A}{B} = \frac{C}{D}$; *to prove that* if $mA > nB$, $mC > nD$; if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$.

(*Dem.*) Since $\frac{A}{B} = \frac{C}{D}$, multiplying (Ax. 1) both by $\frac{m}{n}$, $\frac{mA}{nB} = \frac{mC}{nD}$.

Now, if mA be greater than nB, the first fraction is greater than unity, therefore the second fraction is greater than unity; hence (Ax. 4) $mC > nD$ when $mA > nB$.

Again, if $mA = nB$, the first fraction is equal to unity, hence the second fraction is equal to unity, and therefore $mC = nD$ when $mA = nB$.

Lastly, if $mA < nB$, the first fraction is less than unity, hence the second fraction is less than unity, and therefore $mC < nD$ when $mA < nB$.

PROPOSITION II. THEOREM.

If four magnitudes be such that when equimultiples of the first and third are taken, and equimultiples of the second and fourth, according as the multiple of the first is greater than that of the second, equal to it, or less, so is the multiple of the third greater than that of the fourth, equal to it, or less; the first divided by the second gives the same quotient as the third divided by the fourth.

Given four magnitudes A, B, C, and D, such that if $mA > nB$, $mC > nD$; if $mA = nB$, $mC = nD$; and if $mA < nB$, $mC < nD$; *to prove that* $\dfrac{A}{B} = \dfrac{C}{D}$.

(*Dem.*) Let $\dfrac{A}{B} = \dfrac{C}{E}$; then, by (Prop. I.), if $mA > nB$, $mC > nE$; if $mA = nB$, $mC = nE$; and if $mA < nB$, $mC < nE$; but by the hypothesis, when $mA = nB$, $mC = nD$; and it has been proved that when $mA = nB$, $mC = nE$, therefore when $mC = nD$, mC is also $= nE$; hence $nD = nE$, and $D = E$.

But $\dfrac{A}{B} = \dfrac{C}{E}$, therefore $\dfrac{A}{B} = \dfrac{C}{D}$.

PROPOSITION III. THEOREM. (EUCLID IV.)

If any equimultiples be taken of the antecedents of an analogy, and any equimultiples of the consequents, these multiples, taken in the order of the terms, are proportional.

Given $A : B :: C : D$, and let m and n be any two numbers; *to prove that* $mA : nB :: mC : nD$.

(*Dem.*) Since $A : B :: C : D$, $\dfrac{A}{B} = \dfrac{C}{D}$; multiply both by $\dfrac{m}{n}$, then (Ax. 1) $\dfrac{mA}{nB} = \dfrac{mC}{nD}$; hence, by the definition, $mA : nB :: mC : nD$.

COR.—In the same manner it may be demonstrated that if $A : B :: C : D$, $mA : B :: mC : D$, and $A : nB :: C : nD$. For the first is derived from the proposition by making $n = 1$; and the second by making $m = 1$.

PROPOSITION IV. THEOREM. (PROP. A.)

If four magnitudes be proportionals, they are also proportionals when taken inversely.

Given that $A : B :: C : D$; *to prove that* $B : A :: D : C$.

(*Dem.*) Since $A : B :: C : D$, therefore $\frac{A}{B} = \frac{C}{D}$; hence $1 \div \frac{A}{B} = 1 \div \frac{C}{D}$ (Ax. 2), or $1 \times \frac{B}{A} = 1 \times \frac{D}{C}$, whence $\frac{B}{A} = \frac{D}{C}$; and therefore $B : A :: D : C$, by definition.

PROPOSITION V. THEOREM. (PROP. B.)

If the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second as the third is to the fourth.

Given mA and mB equimultiples of the magnitudes A and B; *to prove that* $mA : A :: mB : B$.

(*Dem.*) $\frac{mA}{A}$ and $\frac{mB}{B}$ are each $= m$, hence $\frac{mA}{A} = \frac{mB}{B}$, therefore $mA : A :: mB : B$, by definition.

Again, since $mA : A :: mB : B$, $A : mA :: B : mB$ (Prop. 4), where A is the same part of mA that B is of mB.

PROPOSITION VI. THEOREM. (PROP. C.)

If the first term of an analogy be a multiple, or a part of the second, the third is the same multiple or part of the fourth.

Given $A : B :: C : D$; and first let A be a multiple of B; *to prove that* C is the same multiple of D; that is, if $A = mB$, $C = mD$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$, and dividing both by m, $\frac{A}{mB} = \frac{C}{mD}$; but, by hypothesis, $\frac{A}{mB} = 1$ (Ax. 4); hence $\frac{C}{mD} = 1$, and therefore $C = mD$.

Next, let A be a part of B, so that $mA = B$; *to prove that* $mC = D$. Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$, and multiplying

both by m, $\frac{mA}{B} = \frac{mC}{D}$; but, by hypothesis, $\frac{mA}{B} = 1$ (Ax. 4);

hence $\frac{mC}{D} = 1$, and therefore $mC = D$.

PROPOSITION VII. THEOREM. (EUCLID VII.)

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

Given A and B two equal magnitudes, and C any other; *to prove that* $A : C :: B : C$.

(*Dem.*) Since $A = B$ by hypothesis, $\frac{A}{C} = \frac{B}{C}$ (Ax. 2), and therefore $A : C :: B : C$.

Again, since $A : C :: B : C$, by inversion (Prop. 4), $C : A :: C : B$.

PROPOSITION VIII. THEOREM. (EUCLID VIII.)

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less than it has to the greater.

Given $A + B$ a magnitude greater than A, and C a third magnitude; *to prove that* $A + B$ has to C a greater ratio than A has to C; and conversely, that C has a greater ratio to A than it has to $A + B$.

(*Dem.*) Since $A + B > A$, $\frac{A + B}{C} > \frac{A}{C}$ (Ax. 5), and therefore $A + B : C > A : C$.

Conversely, since $C = C$, $\frac{C}{A} > \frac{C}{A + B}$, and therefore $C : A > C : A + B$.

PROPOSITION IX. THEOREM. (EUCLID IX.)

Magnitudes which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.

First, *given* $A : C :: B : C$; *to prove that* $A = B$.

(*Dem.*) Since $A : C :: B : C$, $\frac{A}{C} = \frac{B}{C}$, and multiplying both by C, $A = B$.

Second, *given* $C : A :: C : B$; *to prove that* $A = B$.

(*Dem.*) Since $C : A :: C : B$, $A : C :: B : C$ (Prop. 4), and therefore by the first case, $A = B$.

PROPOSITION X. THEOREM. (EUCLID X.)

That magnitude which has a greater ratio than another has to the same magnitude is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude is the less of the two.

(1.) *Given* the ratio of A to C greater than that of B to C; *to prove that* $A > B$.

(*Dem.*) Since, by hypothesis, $\frac{A}{C} > \frac{B}{C}$, by multiplying both by C, $A > B$.

(2.) *Given* that $C : B > C : A$; *to prove that* $B < A$.

(*Dem.*) Since $C : B > C : A$, $\frac{C}{B} > \frac{C}{A}$, dividing both by C, $\frac{1}{B} > \frac{1}{A}$, and multiplying both by the product of A into B gives $A > B$, or $B < A$.

PROPOSITION XI. THEOREM. (EUCLID XI.)

Ratios that are equal to the same ratio are equal to one another.

Given that $A : B :: C : D$, and that $C : D :: E : F$; *to prove that* $A : B :: E : F$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$, and since $C : D :: E : F$, $\frac{C}{D} = \frac{E}{F}$; therefore since $\frac{A}{B}$ and $\frac{E}{F}$ are each equal to $\frac{C}{D}$, therefore $\frac{A}{B} = \frac{E}{F}$, and hence $A : B :: E : F$.

PROPOSITION XII. THEOREM. (EUCLID XII.)

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Given that $A : B :: C : D$, and that $C : D :: E : F$; *to prove that* $A : B :: A + C + E : B + D + F$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$,

and since $C : D :: E : F$, $\frac{C}{D} = \frac{E}{F}$; let each of these quotients or ratios be equal to r; hence $\frac{A}{B} = \frac{C}{D} = \frac{E}{F} = r$.

And since $\frac{A}{B} = r$, $A = rB$ (Ax. 1).

Similarly, ... $C = rD$ ...

and ... $E = rF$ ... ; and adding equals to equals, we obtain $A + C + E = r(B + D + F)$, and dividing both by $B + D + F$, gives $\frac{A + C + E}{B + D + F} = r$, but $r = \frac{A}{B}$;

hence $\frac{A}{B} = \frac{A + C + E}{B + D + F}$, and therefore

$$A : B :: A + C + E : B + D + F.$$

COR.—Since it has been proved that $\frac{A + C + E}{B + D + F} = r$, which is equal to either of the three fractions, and that $A + C + E$ is the sum of all the numerators, and $B + D + F$ is the sum of all the denominators; if any number of fractions be equal, the sum of the numerators divided by the sum of the denominators is a fraction equal in value to either of the given fractions.

PROPOSITION XIII. THEOREM. (EUCLID XIII.)

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth, the first shall have to the second a greater ratio than the fifth has to the sixth.

Given that $A : B :: C : D$, but that $C : D > E : F$; *to prove that* $A : B > E : F$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$; and since $C : D > E : F$, $\frac{C}{D} > \frac{E}{F}$, therefore (Ax. 15, Book I.) $\frac{A}{B} > \frac{E}{F}$, and hence $A : B > E : F$.

PROPOSITION XIV. THEOREM. (EUCLID XIV.)

If the first term of an analogy be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Given $A : B :: C : D$; *to prove that* if $A > C$, $B > D$, $A = C$, $B = D$, and if $A < C$, $B < D$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D} = r$, say,

then $A = rB$, and $C = rD$; therefore

if $A > C$, $rB > rD$, wherefore $B > D$,

if $A = C$, $rB = rD$, ... $B = D$,

and if $A < C$, $rB < rD$, ... $B < D$.

PROPOSITION XV. THEOREM. (EUCLID XV.)

Magnitudes have the same ratio to one another which their equimultiples have.

Given A and B two magnitudes, and m any number; *to prove that* $A : B :: mA : mB$.

(*Dem.*) By the Arithmetical Principles (5) at the beginning of this Book, $\frac{A}{B} = \frac{mA}{mB}$; and therefore $A : B :: mA : mB$.

PROPOSITION XVI. THEOREM. (EUCLID XVI.)

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

Given four magnitudes, A, B, C, and D, of the same kind, such that $A : B :: C : D$; *to prove that* $A : C :: B : D$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$; and multiplying both by $\frac{B}{C}$ (Arith. Prin. 3), $\frac{A}{B} \times \frac{B}{C} = \frac{C}{D} \times \frac{B}{C}$, hence (Arith. Prin. 5), $\frac{A}{C} = \frac{B}{D}$, wherefore $A : C :: B : D$.

PROPOSITION XVII. THEOREM. (EUCLID XVII.)

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if the first, together with the second, have to the second the same ratio which the third, together with the fourth, has to the fourth, the first shall have to the second the same ratio which the third has to the fourth.

Given that $A + B : B :: C + D : D$; *to prove that* $A : B :: C : D$.

(*Dem.*) Since $A + B : B :: C + D : D$, $\frac{A+B}{B} = \frac{C+D}{D}$; therefore $\frac{A}{B} + \frac{B}{B} = \frac{C}{D} + \frac{D}{D}$, but $\frac{B}{B} = \frac{D}{D}$ are each equal to one, and taking away these equals, there remains $\frac{A}{B} = \frac{C}{D}$, and therefore $A : B :: C : D$.

PROPOSITION XVIII. THEOREM. (EUCLID XVIII.)

The terms of an analogy are proportional by composition.

Given that $A : B :: C : D$; *to prove that*, by composition, $A + B : B :: C + D : D$.

(*Dem.*) Since $A : B :: C : D$, $\frac{A}{B} = \frac{C}{D}$; but $\frac{B}{B} = \frac{D}{D}$, and adding these equals to the former, $\frac{A+B}{B} = \frac{C+D}{D}$, and therefore $A + B : B :: C + D : D$.

PROPOSITION XIX. THEOREM. (EUCLID XIX.)

If four magnitudes of the same kind be proportionals, the difference of the antecedents is to the difference of the consequents as either antecedent to its consequent.

Given four magnitudes of the same kind, such that $A : B :: C : D$; *to prove that* if A be greater than C, $A - C : B - D :: A : B$, or as $C : D$.

(*Dem.*) Since the four magnitudes are of the same kind (V. 16), $A : C :: B : D$, therefore $\frac{A}{C} = \frac{B}{D}$, but $\frac{C}{C} = \frac{D}{D}$, and taking

these equals from the former $\frac{A - C}{C} = \frac{B - D}{D}$, hence $A - C : C :: B - D : D$, and alternately (V. 16), $A - C : B - D :: C : D$. But $A : B :: C : D$, therefore (V. 11) $A - C : B - D :: A : B$.

In the same manner it may be demonstrated that if C is greater than A, $C - A : D - B :: A : B$, or as $C : D$.

PROPOSITION XX. THEOREM. (EUCLID D.)

The terms of an analogy are proportional by conversion.

Given that $A : B :: C : D$; *to prove that*, by conversion, $A : A - B :: C : C - D$.

(*Dem.*) Since $A : B :: C : D$, by inversion (Prop. 4), $B : A :: D : C$, therefore $\frac{B}{A} = \frac{D}{C}$, but $\frac{A}{A} = \frac{C}{C}$, from these equals take the former equals, hence $\frac{A - B}{A} = \frac{C - D}{C}$, therefore $A - B : A :: C - D : C$, and again, by inversion (Prop. 4), $A : A - B :: C : C - D$.

PROPOSITION XXI. THEOREM. (EUCLID F.)

The terms of an analogy are proportional by addition.

Given that $A : B :: C : D$; *to prove that* $A : A + B :: C : C + D$.

(*Dem.*) Since $A : B :: C : D$, by inversion (Prop. 4), $B : A :: D : C$, therefore $\frac{B}{A} = \frac{D}{C}$, but $\frac{A}{A} = \frac{C}{C}$, and adding these equals to the former, $\frac{A + B}{A} = \frac{C + D}{C}$, hence $A + B : A :: C + D : C$; and, by inversion (Prop. 4),

$$A : A + B :: C : C + D.$$

PROPOSITION XXII. THEOREM. (EUCLID XXII.)

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio, which the first of the others has to the last.

First, let there be *given* three magnitudes, A, B, and C, and other three, D, E, and F, which, taken two and two in order, have the same ratio; namely, $A:B::D:E$, and $B:C::E:F$; *to prove that* $A:C::D:F$,

(*Dem.*) Since $A:B::D:E$, $\frac{A}{B}=\frac{D}{E}$, but $\frac{B}{C}=\frac{E}{F}$, since $B:C::E:F$; and multiplying the former equals by the latter, $\frac{A}{B}\times\frac{B}{C}=\frac{D}{E}\times\frac{E}{F}$, whence (Arith. Prin. 5), $\frac{A}{C}=\frac{D}{F}$, and therefore $A:C::D:F$.

Again, let there be *given* four magnitudes, A, B, C, and D, and other four, E, F, G, and H; which, taken two and two in order, have the same ratio; namely, $A:B::E:F$, $B:C::F:G$, and $C:D::G:H$; *to prove that* $A:D::E:H$.

(*Dem.*) By the three given proportions, $\frac{A}{B}=\frac{E}{F}$, $\frac{B}{C}=\frac{F}{G}$, and $\frac{C}{D}=\frac{G}{H}$, and taking the product of these equal fractions or ratios, $\frac{A}{B}\times\frac{B}{C}\times\frac{C}{D}=\frac{E}{F}\times\frac{F}{G}\times\frac{G}{H}$, and dividing numerator and denominator of the first side by $B\times C$, and of the second by $F\times G$ (Arith. Prin. 5), $\frac{A}{D}=\frac{E}{H}$, and therefore $A:D::E:H$.

In the same manner the demonstration may be extended to any number of such magnitudes.

PROPOSITION XXIII. THEOREM. (EUCLID XXIII.)

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last.

First, let there be *given* three magnitudes, A, B, and C, and other three, D, E, and F, which, taken two and two in a cross order, have the same ratio; namely, $A:B::E:F$, and $B:C::D:E$; *to prove that* $A:C::D:F$.

(*Dem.*) Since $A : B :: E : F$, and $B : C :: D : E$. $\frac{A}{B} = \frac{E}{F}$, and $\frac{B}{C} = \frac{D}{E}$, and multiplying equals by equals, $\frac{A}{B} \times \frac{B}{C} = \frac{E}{F} \times \frac{D}{E}$, cancelling the common multipliers from numerator and denominator on both sides (Arith. Prin. 5), $\frac{A}{C} = \frac{D}{F}$, and therefore $A : C :: D : F$.

Next, let there be *given* four magnitudes, A, B, C, and D, and other four, E, F, G, and H, which, taken two and two in a cross order, have the same ratio; namely, $A : B :: G : H$, $B : C :: F : G$, and $C : D :: E : F$; *to prove that* $A : D :: E : H$.

(*Dem.*) From the above given proportions, $\frac{A}{B} = \frac{G}{H}$, $\frac{B}{C} = \frac{F}{G}$, and $\frac{C}{D} = \frac{E}{F}$; multiplying equals by equals, $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} = \frac{G}{H} \times \frac{F}{G} \times \frac{E}{F}$; again, cancelling the common factors from the numerator and denominator of both sides (Arith. Prin. 5), $\frac{A}{D} = \frac{E}{H}$, and therefore $A : D :: E : H$.

In the same manner the demonstration may be extended to any number of such magnitudes.

PROPOSITION XXIV. THEOREM. (EUCLID XXIV.)

If the first has to the second the same ratio which the third has to the fourth; and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.

Let the six *given* magnitudes, A, B, C, D, E, and F, be such that $A : B :: C : D$, and $E : B :: F : D$; it is required *to prove that* $A + E : B :: C + F : D$.

(*Dem.*) From the *given* proportions, $\frac{A}{B} = \frac{C}{D}$, and $\frac{E}{B} = \frac{F}{D}$, and adding equals to equals, $\frac{A+E}{B} = \frac{C+F}{D}$; and therefore $A + E : B :: C + F : D$.

PROPOSITION XXV. THEOREM. (EUCLID E.)

Ratios which are compounded of the same or equal ratios are equal to one another.

Given the ratios of A to B, and of B to C, which compound the ratio of A to C, equal, each to each, to the ratios of D to E, and E to F, which compound the ratio of D to F; *to prove that* A : C :: D : F.

(*Dem.*) For, first, if the ratio of A to B be equal to that of D to E, and the ratio of B to C equal to that of E to F, then (V. 22), A : C :: D : F.

And next, if the ratio of A to B be equal to that of E to F, and the ratio of B to C equal to that of D to E, then (V. 23), A : C :: D : F.

In the same manner the proposition may be demonstrated whatever be the number of ratios.

PROPOSITION XXVI. THEOREM. (EUCLID G.)

The terms of a proportion are also proportional by mixing.

Given that A : B :: C : D; *to prove that* A + B : A − B :: C + D : C − D.

(*Dem.*) Since A : B :: C : D, $\frac{A}{B} = \frac{C}{D}$, and $\frac{B}{B} = \frac{D}{D}$, each being equal to unity, then if A $>$ B, adding the second equals to the first, gives $\frac{A+B}{B} = \frac{C+D}{D}$, and then subtracting them, gives $\frac{A-B}{B} = \frac{C-D}{D}$. Again, dividing equals by equals, gives $\frac{A+B}{B} \times \frac{B}{A-B} = \frac{C+D}{D} \times \frac{D}{C-D}$, or $\frac{A+B}{A-B} = \frac{C+D}{C-D}$ (Arith. Prin. 5), and therefore

$$A + B : A - B :: C + D : C - D.$$

In the same manner, if A $<$ B it can be demonstrated that A + B : B − A :: C + D : D − C.

PROPOSITION XXVII. THEOREM.

If four magnitudes be proportional, the sum of the greatest and least is greater than the sum of the other two.

Given that A : B :: C : D, and that A is the greatest term; *to prove that* A + D > B + C.

(*Dem.*) Since A is the greatest, it is greater than either B or C; but (V. 14) when A > C, B > D, and, by alternation (V. 16), A : C :: B : D, therefore (V. 14) since A > B, C > D, hence D is the least; but (V. 20) A : A − B :: C : C − D, and since A > C, A − B > C − D (V. 14), add B + D to both, therefore A + D > B + C.

SIXTH BOOK.

DEFINITIONS.

1. Straight lines are said to be *similarly divided* when their corresponding segments are proportional.

2. A line is said to be cut in a *given ratio* when its segments have that ratio.

3. Straight lines that meet in the same point are called *convergent* or *divergent* lines, according as they are considered to verge towards or from the point of concourse.

4. A straight line is said to be cut in *extreme* and *mean ratio*, when the whole is to the greater segment, as the greater segment is to the less.

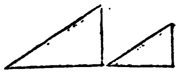
5. A straight line is said to be *harmonically* divided, when it is divided into three segments such, that the whole line is to one extreme segment, as the other extreme segment to the middle segment.

6. Three straight lines are said to be in *harmonical progression* or *harmonical progressionals*, when the first is to the third, as the difference between the first and second is to the difference between the second and third.

7. The second of three lines in this progression is called a *harmonic mean* between the other two; and the third is called a *third harmonical progressional* to the other two.

8. Four divergent lines that cut any line harmonically are called *harmonicals*.

9. *Similar rectilineal figures* are those which have their several angles equal, each to each, and the sides about the equal angles proportionals.



10. Two sides of one figure are said to be *reciprocally proportional* to two sides of another, when one of the sides of the first is

to one of the sides of the other, as the remaining side of the other is to the remaining side of the first.

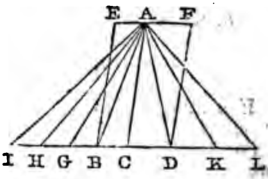
11. A *lune* is a figure contained by two arcs of different circles.

PROPOSITION I. THEOREM.

Triangles and parallelograms of the same altitude are one to another as their bases.

Given the triangles ABC, ACD, and the parallelograms EC, CF, having the same altitude; namely, the perpendicular drawn from the point A to BD; *to prove that* as the base BC is to the base CD, so is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

(*Const.*) Produce BD both ways to the points I and L, and take any number of straight lines BG, GH, HI, each equal to the base BC, and DK, KL, any number of them, each equal to the base CD; and join AI, AH, AG, AK, and AL. (*Dem.*) Then, because CB, BG, GH, and HI are all equal, the triangles AIH, AHG, AGB, ABC are all equal (I. 38); therefore, whatever multiple the base IC is of the base BC, the same multiple is the triangle AIC of the triangle ABC. For the same reason, whatever multiple the base LC is of the base CD, the same multiple is the triangle ALC of the triangle ADC. And if the base IC be equal to the base CL, the triangle AIC is also equal to the triangle ALC; and if the base IC be greater than the base CL, the triangle AIC is likewise greater than the triangle ALC; and if less, less.



Therefore, since there are four magnitudes; namely, the two bases BC, CD, and the two triangles ABC, ACD; and of the base BC and the triangle ABC, the first and third, any equimultiples whatever have been taken; namely, the base IC and triangle AIC; and of the base CD and triangle ACD, the second and fourth, there have been taken any equimultiples whatever; namely, the base CL and triangle ALC; and since it has been shewn that if the base IC be greater than the base CL, the triangle AIC is greater than the triangle ALC; and if equal, equal; and if less, less. Therefore (V. Def. 10), as the base BC is to the base CD, so is the triangle ABC to the triangle ACD.

Again, because the parallelogram CE is double of the triangle ABC (I. 41), and the parallelogram CF double of the triangle ACD, and that magnitudes have the same ratio which their

equimultiples have (V. 15); as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF. And because it has been shewn that as the base BC is to the base CD, so is the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, so is the parallelogram EC to the parallelogram CF; therefore, as the base BC is to the base CD, so is the parallelogram EC to the parallelogram CF (V. 11).

COR. 1.—From this it is plain, that triangles and parallelograms that have equal altitudes are one to another as their bases.

For, if the figures be placed so as to have their bases in the same straight line, then the straight line which joins the vertices is parallel to that in which their bases are (I. 34, Cor. 2). Then, if the same construction be made as in the proposition, the demonstration will be the same.

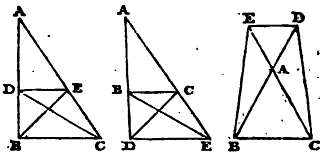
COR. 2.—Rectangles, and hence also parallelograms and triangles, having equal bases, are to one another as their altitudes.

PROPOSITION II. THEOREM.

If a straight line be drawn parallel to one of the sides of a triangle, it shall cut the other sides, or the other sides produced proportionally; and if the sides, or the sides produced, be cut proportionally, the straight line which joins the points of section shall be parallel to the remaining side of the triangle.

Given DE a line drawn parallel to BC, one of the sides of the triangle ABC; *to prove that* BD is to DA, as CE to EA.

(*Const.*) Join BE and CD; (*Dem.*) then the triangle BDE is equal to the triangle CDE (I. 37), because they are on the same base DE, and between the same parallels DE and BC. But ADE is another triangle, and equal magnitudes have to the same the same ratio (V. 7); therefore, as the triangle BDE to the triangle ADE, so is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, so is BD to DA (VI. 1); because, having the same altitude, namely, the perpendicular drawn from the point E to AB, they are to one another as their bases; and for the same reason, as



the triangle CDE to the triangle ADE, so is CE to EA. Therefore as BD to DA, so is CE to EA (V. 11).

Next, let it be *given* that the sides AB and AC, of the triangle ABC, or these sides produced, are cut proportionally in the points D and E; that is, so that BD is to DA as CE to EA, and join DE; *to prove that* DE is parallel to BC.

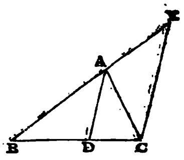
The same construction being made, because as BD to DA, so is CE to EA; and as BD to DA, so is the triangle BDE to the triangle ADE (VI. 1); and as CE to EA so is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE (V. 11); that is, the triangles BDE and CDE have the same ratio to the triangle ADE; and therefore the triangle BDE is equal to the triangle CDE (V. 9). And they are on the same base DE; but equal triangles on the same base are between the same parallels (I. 39); therefore DE is parallel to BC.

PROPOSITION III. THEOREM.

If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base shall have the same ratio which the other sides of the triangle have to one another; and if the segments of the base have the same ratio which the other sides of the triangle have to one another, the straight line drawn from the vertex to the point of section bisects the vertical angle.

Let it be *given* that the angle BAC of the triangle ABC is divided into two equal angles by the straight line AD; *to prove that* BD is to DC as BA to AC.

(*Const.*) Through the point C draw CE parallel to DA (I. 31), and let BA produced meet CE in E. (*Dem.*) Because the straight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD (I. 29).



But CAD is given equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, because the straight line BAE meets the parallels AD, EC, the exterior angle BAD is equal to the interior and opposite angle AEC. But the angle ACE has been proved equal to the angle BAD; therefore also ACE is equal to the angle AEC, and consequently the side AE is equal to the side AC (I. 6). And because AD is drawn parallel to EC one of the sides of the triangle BCE, BD is to DC, as BA to AE (VI. 2). But AE is equal to AC; therefore as BD to DC, so is BA to AC (V. 7).

Next, let it be *given* that BD is to DC as BA to AC, and join AD; *to prove that* the angle BAC is divided into two equal angles by the straight line AD.

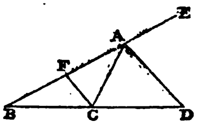
The same construction being made; (*Dem.*) because BD is to DC as BA to AC; and BD is to DC as BA to AE (VI. 2), because AD is parallel to EC; therefore AB is to AC as BA to AE (V. 11); consequently AC is equal to AE (V. 9), and the angle AEC is therefore equal to the angle ACE (I. 5). But the angle AEC is equal to the external and opposite angle BAD; and the angle ACE is equal to the alternate angle CAD. Wherefore, also, the angle BAD is equal to the angle CAD; therefore the angle BAC is cut into two equal angles by the straight line AD.

PROPOSITION A. THEOREM.

If the exterior angle of a triangle be bisected by a straight line which also cuts the base produced, the segments between the bisecting line and the extremities of the base have the same ratio which the other sides of the triangle have to one another; and if the segments of the base produced have the same ratio which the other sides of the triangle have, the straight line drawn from the vertex to the point of section bisects the exterior angle of the triangle.

Let it be *given* that the exterior angle CAE of any triangle ABC is divided into two equal angles by the straight line AD which meets the base produced in D; *to prove that* BD is to DC as BA to AC.

(*Const.*) Through C draw CF parallel to AD (I. 31). (*Dem.*) And because the straight line AC meets the parallels AD, FC, the angle ACF is equal to the alternate angle CAD (I. 29); but CAD is equal to the angle DAE (Hyp.); therefore also DAE is equal to the angle ACF. Again, because the straight line FAE meets the parallels AD, FC, the exterior angle DAE is equal to the interior and opposite angle CFA. But the angle ACF has been proved to be equal to the angle DAE;



therefore also the angle ACF is equal to the angle CFA; and consequently the side AF is equal to the side AC (I. 6); and because AD is parallel to FC, a side of the triangle BCF; BD is to DC as BA to AF (VI. 2); but AF is equal to AC; therefore BD is to DC as BA to AC.

Next, let it be *given* that BD is to DC as BA to AC, and join AD; *to prove that* the angle CAD is equal to the angle DAE.

The same construction being made, (*Dem.*) because BD is to DC as BA to AC; and also BD to DC as BA to AF; therefore BA is to AC as BA to AF (V. 11); wherefore AC is equal to AF (V. 9), and the angle AFC equal to the angle

ACF (I. 5). But the angle AFC is equal to the exterior angle EAD, and the angle ACF to the alternate angle CAD; therefore also EAD is equal to the angle CAD.

COR.—The two lines that bisect the vertical angle and its adjacent exterior angle cut the base produced harmonically; or the base is a harmonic mean between its greater internal and external segments.

Schol. 1.—This and the last propositions may be enunciated thus: If the vertical angle of a triangle and its adjacent angle be bisected by lines cutting the base, it will be cut internally and externally in the ratio of the two sides.

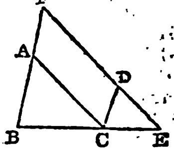
Schol. 2.—By means of these two propositions, it is proved in optics that the axis of a pencil of rays incident on a spherical mirror is divided harmonically by the radiant point, the geometrical focus of reflected rays, and the centre and surface of the reflector. It is also found that the lengths of three musical strings of the same thickness, material, and texture, and under the same tension, that produce any note, its fifth, and octave, are in *harmonical progression;* hence the origin of the term. It is believed that Pythagoras first observed this relation of musical strings.

PROPOSITION IV. THEOREM.

The sides about the equal angles of equiangular triangles are proportionals; and those which are opposite to the equal angles are homologous sides; that is, are the antecedents or consequents of the ratios.

Given ABC, DCE, two equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and consequently the angle BAC equal to the angle CDE (I. 32); *to prove that* the sides about the equal angles of the triangles ABC, DCE are proportionals; and those sides are homologous which are opposite to the equal angles.

(*Const.*) Let the triangle DCE be placed so that its side CE may be contiguous to BC, and in the same straight line with it; and because the angles ABC, ACB are together less than two right angles (I. 17), ABC and DEC, which is equal to ACB, are also less than two right angles; wherefore BA and ED, being produced, shall meet (I. 29, Cor.); let them be produced and meet in the point F. (*Dem.*) And because the angle ABC is equal to the angle DCE, BF is parallel to CD (I. 28). Again, because the angle ACB is equal to the angle DEC, AC is parallel to FE; therefore FACD is a parallelogram;



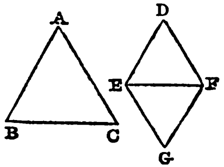
and consequently AF is equal to CD, and AC to FD (I. 34); and because AC is parallel to FE, one of the sides of the triangle FBE, BA : AF :: BC : CE (VI. 2); but AF is equal to CD; therefore (V. 7), BA : CD :: BC : CE; and alternately, AB : BC :: DC : CE. Again, because CD is parallel to BF, BC : CE :: FD : DE; but FD is equal to AC; therefore, BC : CE :: AC : DE; and alternately, BC : CA :: CE : ED. Therefore, because it has been proved that AB : BC :: DC : CE; and BC : CA :: CE : ED, by equality (V. 22), BA : AC :: CD : DE.

PROPOSITION V. THEOREM.

If the sides of two triangles, about each of their angles, be proportionals, the triangles shall be equiangular, and have their angles equal which are opposite to the homologous sides.

Given the triangles ABC and DEF having their sides proportionals, so that AB is to BC as DE to EF; and BC to CA as EF to FD; and consequently, by equality, BA to AC as ED to DF; *to prove that* the triangle ABC is equiangular to the triangle DEF, and their equal angles are opposite to the homologous sides; namely, the angle ABC being equal to the angle DEF, and BCA to EFD, and also BAC to EDF.

(*Const.*) At the points E and F, in the straight line EF, make (I. 23) the angle FEG equal to the angle ABC, and the angle EFG equal to BCA; (*Dem.*) wherefore the remaining angle BAC is equal to the remaining angle EGF (I. 32), and the triangle ABC is therefore equiangular to the triangle GEF; and consequently they have their sides opposite to the equal angles proportionals (VI. 4); wherefore AB is to BC as GE to EF; but AB is to BC as DE to EF; therefore DE is to EF as GE to EF (V. 11); therefore DE and GE have the same ratio to EF, and consequently are equal (V. 9); for the same reason, DF is equal to FG;



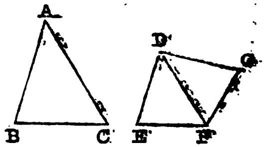
and because, in the triangles DEF, GEF, DE is equal to EG, and EF common, and also the base DF equal to the base GF; therefore (I. 8) the angle DEF is equal to the angle GEF; in the same manner it may be proved that the angle DFE is equal to GFE, and EDF to EGF; and because the angle DEF is equal to the angle GEF, and GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF; for the same reason, the angle ACB is equal to the angle DFE, and the angle at A to the angle at D; therefore the triangle ABC is equiangular to the triangle DEF.

PROPOSITION VI. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals; the triangles shall be equiangular, and shall have those angles equal which are opposite to the homologous sides.

Given the triangles ABC and DEF having the angle BAC in the one equal to the angle EDF in the other; and the sides about those angles proportionals; that is, BA to AC, as ED to DF; *to prove that* the triangles ABC and DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

(*Const.*) At the points D and F, in the straight line DF, make (I. 23) the angle FDG equal to either of the angles BAC or EDF; and the angle DFG equal to the angle ACB; (*Dem.*) wherefore the remaining angle at B is equal to the remaining one at G (I. 32), and consequently the triangle ABC is equiangular to the triangle DGF; and therefore BA is to AC as GD to DF (VI. 4); but, by the hypothesis, BA is to AC as ED to DF; therefore ED is to DF as GD to DF (V. 11); wherefore ED is equal to DG (V. 9); and DF is common to the two triangles EDF and GDF; therefore the two sides ED and DF are equal to the two sides GD and DF; but the angle EDF is also equal to the angle GDF; therefore (I. 4) the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE; and the angle BAC is equal to the angle EDF (Hyp.); wherefore also the remaining angle at B is equal to the remaining angle at E; therefore the triangle ABC is equiangular to the triangle DEF.



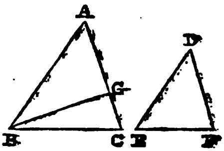
PROPOSITION VII. THEOREM.

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals, then if each of the remaining angles be either less, or not less, than a right angle, the triangles shall be equiangular, and have those angles equal about which the sides are proportionals.

Given the two triangles ABC and DEF having one angle in the one equal to one angle in the other; namely, the angle BAC to the angle EDF, and the sides about two other angles ABC and DEF, proportionals, so that AB is to BC as DE to

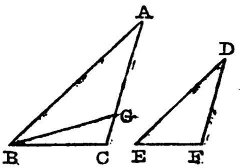
EF; and in the first case, let each of the remaining angles at C and F be less than a right angle; *to prove that* the triangle ABC is equiangular to the triangle DEF; that is, the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.

(*Const.*) For, if the angles ABC and DEF be not equal, one of them is greater than the other. Let ABC be the greater; and at the point B, in the straight line AB, make the angle ABG equal to the angle DEF (I. 23). (*Dem.*) And because the angle at A is equal to the angle at D, and the angle ABG to the angle DEF; the remaining angle AGB is equal to the remaining angle DFE (I. 32); therefore the triangle ABG is equiangular to the triangle DEF; wherefore (VI. 4) AB is to BG as DE to EF; but DE is to EF, by hypothesis, as AB to BC; therefore AB is to BC as AB to BG (V. 11); and because AB has the same ratio to each of the lines BC and BG; BC is equal to BG (V. 9), and therefore the angle BGC is equal to the angle BCG (I. 6); but the angle BCG is, by hypothesis, less than a right angle; therefore also the angle BGC is less than a right angle, and the adjacent angle AGB must be greater than a right angle (I. 13). But it was proved that the angle AGB is equal to the angle at F; therefore the angle at F is greater than a right angle; but, by the hypothesis, it is less than a right angle, which is absurd; therefore the angles ABC and DEF are not unequal; that is, they are equal; and the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F; therefore the triangle ABC is equiangular to the triangle DEF.



Next, let each of the angles at C and F be *given* not less than a right angle; *to prove that* the triangle ABC is also in this case equiangular to the triangle DEF.

The same construction being made, (*Dem.*) it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC; but the angle at C is not less than a right angle; therefore the angle BGC is not less than a right angle; wherefore two angles of the triangle BGC are together not less than two right angles, which is impossible (I. 17); and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the first case.

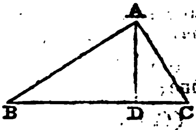


PROPOSITION VIII. THEOREM.

In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side of it are similar to the whole triangle, and to one another.

Given ABC a right-angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the base BC; *to prove that* the triangles ABD and ADC are each similar to the whole triangle ABC, and to one another.

(*Dem.*) Because the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC and ABD; the remaining angle ACB is equal to the remaining angle BAD (I. 32); therefore the triangle ABC is equiangular to the triangle ABD, and the sides about their equal angles are proportionals (VI. 4); wherefore the triangles are similar (VI. Def. 9); in the same manner, it may be demonstrated that the triangle ADC is equiangular and similar to the triangle ABC; and the triangles ABD and ADC, being both equiangular and similar to ABC, are equiangular and similar to each other.



COR.—From this it is manifest that the perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base; and also that each of the sides is a mean proportional between the base, and its segment adjacent to that side. For in the triangles BDA and ADC, BD : DA :: DA : DC (VI. 4); and in the triangles ABC and DBA, BC : BA :: BA : BD; and in the triangles ABC and ACD, BC : CA :: CA : CD.

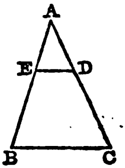
PROPOSITION IX. PROBLEM.

From a given straight line to cut off any part required; that is, a part of which the line shall be a given multiple.

Given the straight line AB; *it is required* to cut off from it a part of which AB shall be a given multiple.

(*Const.*) From the point A draw a straight line AC making any angle with AB; and in AC take any point D, and take AC the same multiple of AD that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it; then AE is the part required to be cut off.

(*Dem.*) Because ED is parallel to one of the sides of the triangle ABC, namely, to BC; CD is to DA as BE to EA (VI. 2); and, by composition (V. 18), CA is to AD as BA to AE. But CA is a multiple of AD; therefore (V. c) BA is the same multiple of AE. Whatever part, therefore, AD is of AC, AE is the same part of AB; wherefore, from the straight line AB the part AE is cut off, which is the part required.

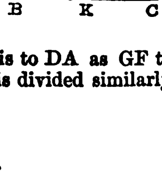


PROPOSITION X. PROBLEM.

To divide a given straight line similarly to a given divided straight line; that is, into parts that shall have the same ratios to one another which the parts of the divided given straight line have.

Given AB the straight line to be divided, and AC the divided line; *it is required* to divide AB similarly to AC.

(*Const.*) Let AC be divided in the points D, E; and let AB and AC be placed so as to contain any angle, and join BC, and through the points D and E draw DF and EG parallel to BC (I. 31); and through D draw DHK parallel to AB; (*Dem.*) therefore each of the figures FH and HB is a parallelogram; wherefore DH is equal to FG, and HK to GB (I. 34). And because HE is parallel to KC, one of the sides of the triangle DKC, CE is to ED as KH to HD (VI. 2). But KH is equal to BG, and HD to GF; therefore CE is to ED as BG to GF. Again, because FD is parallel to EG, one of the sides of the triangle AGE, ED is to DA as GF to FA. But it has been proved that CE is to ED as BG to GF; and ED is to DA as GF to FA; therefore the given straight line AB is divided similarly to AC.



PROPOSITION XI. PROBLEM.

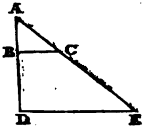
To find a third proportional to two given straight lines.

Given the two straight lines AB and AC, and let them be placed so as to contain any angle; *it is required* to find a third proportional to AB and AC.

(*Const.*) Produce AB and AC; and make BD equal to

AC; and having joined BC, through D, draw DE parallel to it (I. 31).

(*Dem.*) Because BC is parallel to DE, a side of the triangle ADE, AB is to BD as AC to CE (VI. 2). But BD is equal to AC; therefore AB is to AC as AC to CE. Wherefore to the two given straight lines AB and AC, a third proportional CE is found.

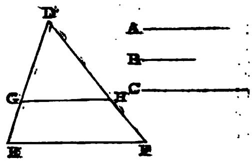


PROPOSITION XII. PROBLEM.

To find a fourth proportional to three given straight lines.

Given the three straight lines A, B, and C; *it is required* to find a fourth proportional to A, B, and C.

(*Const.*) Take two straight lines DE and DF, containing any angle EDF; and upon these make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel to it through the point E (I. 31); (*Dem.*) and because GH is parallel to EF, one of the sides of the triangle DEF; DG is to GE as DH to HF (VI. 2); but DG is equal to A, GE to B, and DH to C; therefore A is to B as C to HF. Wherefore to the three given straight lines A, B, and C, a fourth proportional HF is found.



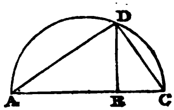
PROPOSITION XIII. PROBLEM.

To find a mean proportional between two given straight lines.

Given the two straight lines AB and BC; *it is required* to find a mean proportional between them.

(*Const.*) Place AB and BC in a straight line, and upon AC describe the semicircle ADC, and (I. 11) from the point B draw BD at right angles at AC, and join AD, DC.

(*Dem.*) Because the angle ADC is in a semicircle, it is a right angle (III. 31), and because in the right-angled triangle ADC, DB is drawn from the right angle perpendicular to the base, DB is a mean proportional between AB and BC, the segments of the base (VI. 8, Cor.); therefore between the two given straight lines AB and BC a mean proportional DB is found.

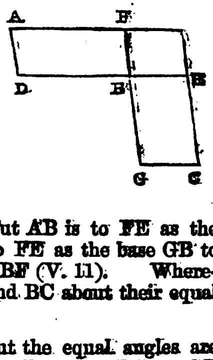


COR.—If AC and CB were the two given lines, by (Prop. 8, Cor.), DC is a mean proportional between them; or if AC and AB were the two given lines, AD is a mean proportional between them. Hence, instead of placing the two straight lines in one and the same straight line, from the greater a part might be cut off equal to the less, and the construction effected as above.

PROPOSITION XIV. THEOREM.

Equal parallelograms which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and parallelograms that have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Given two equal parallelograms AB and BC, which have the angles at B equal, and let the sides DB and BE be placed in the same straight line; wherefore also FB and BG are in one straight line (I. 14); *to prove that* the sides of the parallelograms AB and BC, about the equal angles, are reciprocally proportional; that is, DB is to BE as GB to BF.



(*Const.*) Complete the parallelogram FE. (*Dem.*) And because the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE as BC to FE (V. 7). But AB is to FE as the base DB to BE (VI. 1); and BC is to FE as the base GB to BF; therefore DB is to BE as GB to BF (V. 11). Wherefore the sides of the parallelograms AB and BC about their equal angles are reciprocally proportional.

Next let it be *given* that the sides about the equal angles are reciprocally proportional; namely, that DB is to BE as GB to BF; *to prove that* the parallelogram AB is equal to the parallelogram BC.

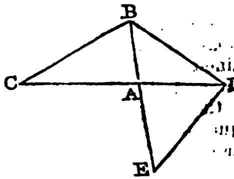
(*Dem.*) Because DB is to BE as GB to BF; and DB is to BE as the parallelogram AB to the parallelogram FE; and GB is to BF as the parallelogram BC to the parallelogram FE; therefore AB is to FE as BC to FE. Wherefore the parallelogram AB is equal to the parallelogram BC (V. 9).

PROPOSITION XV. THEOREM.

Equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional; and triangles which have one angle in the one equal to one angle in the other, and their sides about the equal angles reciprocally proportional, are equal to one another.

Given two equal triangles ABC and ADE, which have the angle BAC equal to the angle DAE; *to prove that* the sides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD as EA to AB.

(*Const.*) Let the triangles be placed so that their sides CA and AD be in one straight line; wherefore also EA and AB are in one straight line (I. 14); join BD. (*Dem.*) Because the triangle ABC is equal to the triangle ADE, and that ABD is another triangle; therefore the triangle CAB is to the triangle BAD as the triangle EAD to the triangle DAB (V. 7). But triangle CAB is to triangle BAD as the base CA to AD (VI. 1); and triangle EAD is to triangle DAB as the base EA to AB; therefore CA is to AD as EA to AB (V. 11); wherefore the sides of the triangles ABC and ADE about the equal angles are reciprocally proportional.



Next, let it be *given* that the sides of the triangles ABC and ADE about the equal angles are reciprocally proportional; namely, CA to AD as EA to AB; *to prove that* the triangle ABC is equal to the triangle ADE.

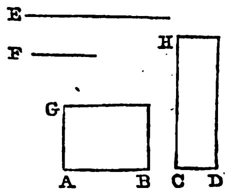
(*Const.*) Having joined BD as before; (*Dem.*) because CA is to AD as EA to AB; and CA is to AD as triangle ABC to triangle BAD (VI. 1); and as EA to AB, so is triangle EAD to triangle BAD; therefore (V. 11), as triangle BAC to triangle BAD, so is triangle EAD to triangle BAD; that is, the triangles BAC and EAD have the same ratio to the triangle BAD. Wherefore the triangle BAC is equal to the triangle ADE (V. 9).

PROPOSITION XVI. THEOREM.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportionals.

Given that the four straight lines AB, CD, E, and F are proportionals; namely, as AB to CD, so is E to F; *to prove that* the rectangle contained by AB and F is equal to the rectangle contained by CD and E.

(*Const.*) From the points A and C, draw AG and CH at right angles to AB and CD (I. 11); and make AG equal to F, and CH equal to E, and complete the parallelograms BG and DH. (*Dem.*) Because AB is to CD as E to F; and that E is equal to CH, and F to AG; AB is to CD as CH to AG (V. 7); therefore the sides of the parallelograms BG and DH about the equal angles are reciprocally proportional; but parallelograms which have their sides about equal angles reciprocally proportional, are equal to one another (VI. 14); therefore the parallelogram BG is equal to the parallelogram DH; and the parallelogram BG is the rectangle contained by the straight lines AB and F; because AG is equal to F; and the parallelogram DH is the rectangle contained by CD and E; because CH is equal to E; therefore the rectangle contained by the straight lines AB and F is equal to that which is contained by CD and E.



Next, let the rectangle contained by the straight lines AB and F be *given* equal to that which is contained by CD and E; *to prove that* these four lines are proportionals; namely, AB is to CD as E to F.

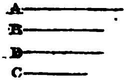
The same construction being made. (*Dem.*) Because the rectangle contained by the straight lines AB and F is equal to that which is contained by CD and E, and that the rectangle BG is contained by AB and F, because AG is equal to F; and the rectangle DH by CD and E, because CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular; but the sides about the equal angles of equal parallelograms are reciprocally proportional (VI. 14); wherefore, as AB to CD, so is CH to AG; and CH is equal to E, and AG to F; therefore AB is to CD as E to F.

PROPOSITION XVII. THEOREM.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean; and if the rectangle contained by the extremes be equal to the square on the mean, the three straight lines are proportionals.

Given that the three straight lines A, B, and C are proportionals; namely, as A to B, so is B to C; *to prove that* the rectangle contained by A and C is equal to the square on B.

(*Const.*) Take D equal to B; (*Dem.*) and because A is to B as B to C, and that B is equal to D; A is to B as D to C; but if four straight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means (VI. 16); therefore the rectangle contained by A and C is equal to that contained by B and D; but the rectangle contained by B and D is the square on B, because B is equal to D; therefore the rectangle contained by A and C is equal to the square on B.



Next, let the rectangle contained by A and C be *given* equal to the square on B; *to prove that* A is to B as B to C.

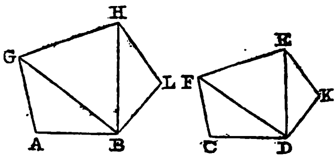
The same construction being made, (*Dem.*) because the rectangle contained by A and C is equal to the square on B, and the square on B is equal to the rectangle contained by B and D, because B is equal to D; therefore the rectangle contained by A and C is equal to that contained by B and D; but if the rectangle contained by the extremes be equal to that contained by the means, the four straight lines are proportionals; therefore A is to B as D to C; but B is equal to D; wherefore A is to B as B to C.

PROPOSITION XVIII. PROBLEM.

Upon a given straight line to describe a rectilineal figure similar, and similarly situated, to a given rectilineal figure.

Let AB be the *given* straight line, and CDEF the *given* rectilineal figure of four sides; *it is required* upon the given straight line AB to describe a rectilineal figure similar, and similarly situated, to CDEF.

(*Const.*) Join DF, and at the points A and B, in the straight line AB, make (I. 23) the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle CFD is equal to the remaining angle AGB (I. 32); wherefore the triangle FCD is equiangular to the triangle GAB. Again, at the points G and B, in the straight line GB, make (I. 23) the angle BGH equal to the angle DFE, and the angle GBH equal to



FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH. (*Dem.*) Then, because the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE; for the same reason, the angle ABH is equal to the angle CDE; also the angle at A is equal to the angle at C, and the angle GHB to FED; therefore the rectilineal figure ABHG is equiangular to CDEF; but likewise these figures have their sides about the equal angles proportionals; because the triangles GAB and FCD, being equiangular, BA is to AG as DC to CF (VI. 4); and because AG is to GB as CF to FD; and as GB to GH, so is FD to FE (VI. 4); therefore (V. 22) AG is to GH as CF to FE; in the same manner, it may be proved that AB is to BH as CD to DE; and GH is to HB as FE to ED (VI. 4); wherefore, because the rectilineal figures ABHG, CDEF are equiangular, and have their sides about the equal angles proportionals, they are similar to one another (VI. Def. 9).

Again, let AB be a *given* straight line, *it is required to* describe a rectilineal figure upon it similar, and similarly situated, to the *given* rectilineal figure CDKEF.

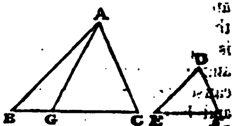
(*Const.*) Join DE, and upon the given straight line AB describe the rectilineal figure ABHG similar, and similarly situated, to the quadrilateral figure CDEF, by the former case; and at the points B and H, in the straight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L. (*Dem.*) And because the figures ABHG, CDEF are similar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK; for a like reason, the angle ABL is equal to the angle CDK; therefore the five-sided figures AGHLB and CFEKD are equiangular; and because the figures AGHB and CFED are similar, GH is to HB as FE to ED; and as HB to HL, so is ED to EK (VI. 4); therefore (V. 22) GH is to HL as FE to EK; for the same reason, AB is to BL as CD to DK; and BL is to LH as DK to KE, because the triangles BLH and DKE are equiangular; therefore, because the five-sided figures AGHLB and CFEKD are equiangular, and have their sides about the equal angles proportionals, they are similar to one another; and in the same manner a rectilineal figure may be described upon a given straight line, similar to one of six or more sides.

PROPOSITION XIX. THEOREM.

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Given that ABC and DEF are similar triangles, having the angle B equal to the angle E, and AB to BC as DE to EF, so that the side BC is homologous to EF (V. Def. 20); *to prove that* the triangle ABC is to the triangle DEF in the duplicate ratio of BC to EF.

(*Const.*) Take BG a third proportional to BC and EF (VI. 11), so that BC is to EF as EF to BG, and join GA. (*Dem.*) Then because AB is to BC as DE to EF; alternately (V. 16) AB is to DE as BC to EF. But as BC to EF, so is EF to BG; therefore (V. 11) AB is to DE as EF to BG. Wherefore the sides of the triangles ABG and DEF, which are about the equal angles, are reciprocally proportional. But triangles which have the sides about two equal angles reciprocally proportional are equal to one another (VI. 15);



therefore the triangle ABG is equal to the triangle DEF; and because BC is to EF as EF to BG; and that if three straight lines be proportionals, the first has to the third the duplicate ratio of that which it has to the second; BC therefore has to BG the duplicate ratio of that which BC has to EF. But as BC to BG, so is the triangle ABC to the triangle ABG (VI. 1). Therefore the triangle ABC is to the triangle ABG in the duplicate ratio of BC to EF. But the triangle ABG is equal to the triangle DEF; wherefore also the triangle ABC is to the triangle DEF in the duplicate ratio of BC to EF.

COR.—From this it is manifest, that if three straight lines be proportionals, as the first is to the third, so is any triangle upon the first to a similar, and similarly described triangle upon the second.

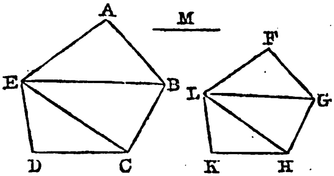
PROPOSITION XX. THEOREM.

Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous sides have.

Given that ABCDE and FGHKL are similar polygons, and that AB is the homologous side to FG; *to prove that* the polygons ABCDE and FGHKL may be divided into the same number of similar triangles, whereof each to each has the same

ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the side FG.

(*Const.*) Join BE, EC, GL, and LH. (*Dem.*) And because the polygon ABCDE is similar to the polygon FGHKL, the angle BAE is equal to the angle GFL; and BA is to AE as GF to FL (VI. Def. 9); wherefore, because the triangles ABE and FGL have an angle in one equal to an angle in the other, and their sides about these equal angles proportionals, the triangle ABE is equiangular (VI. 6), and therefore similar to the triangle FGL (VI. 4); wherefore the angle ABE is equal to the angle FGL. And because the polygons are similar, the whole angle ABC is equal to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH. And because the triangles ABE and FGL are similar, EB is to BA as LG to GF.



And also, because the polygons are similar, AB is to BC as FG to GH; therefore (V. 22) EB is to BC as LG to GH; that is, the sides about the equal angles EBC and LGH are proportionals; therefore the triangle EBC is equiangular to the triangle LGH, and similar to it. For the same reason, the triangle ECD likewise is similar to the triangle LHK; therefore the similar polygons ABCDE, FGHKL are divided into the same number of similar triangles.

Also these triangles have, each to each, the same ratio which the polygons have to one another, the antecedents being ABE, EBC, and ECD, and the consequents FGL, LGH, and LHK. And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the side AB has to the homologous side FG.

Because the triangle ABE is similar to the triangle FGL, ABE has to FGL the duplicate ratio of that which the side BE has to the side GL (VI. 19). For the same reason, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL; therefore as the triangle ABE to the triangle FGL, so is the triangle BEC to the triangle GLH (V. 11). Again, because the triangle EBC is similar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the side EC has to the side LH. For the same reason, the triangle ECD has to the triangle LHK the duplicate ratio of that which EC has to LH; as therefore the triangle EBC to the triangle LGH, so is the triangle

ECD to the triangle LHK. But it has been proved that the triangle EBC is likewise to the triangle LGH as the triangle ABE to the triangle FGL. Therefore as the triangle ABE is to the triangle FGL, so is triangle EBC to triangle LGH, and triangle ECD to triangle LHK; and therefore as one of the antecedents to one of the consequents, so are all the antecedents to all the consequents (V. 12). Wherefore as the triangle ABE to the triangle FGL, so is the polygon ABCDE to the polygon FGHKL; but the triangle ABE has to the triangle FGL the duplicate ratio of that which the side AB has to the homologous side FG. Therefore also the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous side FG.

Cor. 1.—In like manner it may be proved that similar four-sided figures, or of any number of sides, are one to another in the duplicate ratio of their homologous sides, and it has already been proved in triangles; therefore universally similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.

Cor. 2.—If to AB, FG, two of the homologous sides, a third proportional M be taken, AB has (V. Def. 18) to M the duplicate ratio of that which AB has to FG. But the four-sided figure or polygon upon AB has to the four-sided figure or polygon upon FG, likewise the duplicate ratio of that which AB has to FG; therefore as AB is to M, so is the figure upon AB to the figure upon FG, which was also proved in triangles (VI. 19, Cor.). Therefore, universally, it is manifest that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure upon the first, to a similar and similarly described rectilineal figure upon the second.

Cor. 3.—Because all squares are similar figures, the ratio of any two squares to one another is the same with the duplicate ratio of their sides; and hence also any two similar rectilineal figures are to one another as the squares on their homologous sides.

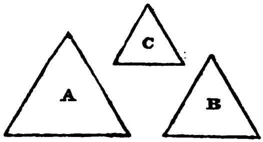
PROPOSITION XXI. THEOREM.

Rectilineal figures which are similar to the same rectilineal figure, are also similar to one another.

Given that each of the rectilineal figures A and B are similar to the rectilineal figure C; *to prove that* the figure A is similar to the figure B.

(Dem.) Because A is similar to C, they are equiangular, and

also have their sides about the equal angles proportionals (VI. Def. 9). Again, because B is similar to C, they are equiangular, and have their sides about the equal angles proportionals; therefore the figures A and B are each of them equiangular to C, and have the sides about the equal angles of each of them and of C proportionals; wherefore the rectilineal figures A and B are equiangular, and have their sides about the equal angles proportionals (V. 11); therefore A is similar to B.

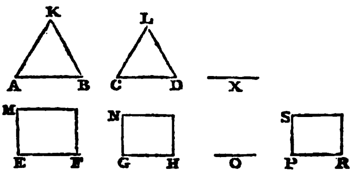


PROPOSITION XXII. THEOREM.

If four straight lines be proportionals, the similar rectilineal figures similarly described upon them shall also be proportionals; and if the similar rectilineal figures similarly described upon four straight lines be proportionals, those straight lines shall be proportionals.

Let the four straight lines AB, CD, EF, and GH be *given* proportionals; namely, AB to CD as EF to GH, and upon AB and CD let the similar rectilineal figures KAB and LCD be similarly described; and upon EF and GH the similar rectilineal figures MF and NH, in like manner; *to prove that* the rectilineal figure KAB is to LCD as MF to NH.

(*Const.*) To AB and CD take a third proportional X (VI. 11); and to EF and GH a third proportional O. (*Dem.*) And because AB is to CD as EF to GH, and that CD is to X as GH to O (V. 11); wherefore (V. 22) AB is to X as EF to O; but as AB to X, so is the rectilineal KAB to the rectilineal LCD (VI. 20, Cor. 2), and as EF to O, so is the rectilineal MF to the rectilineal NH; therefore as KAB to LCD, so is MF to NH.



Again, let the rectilineal KAB be to LCD as MF to NH; *to prove that* the straight line AB is to CD as EF to GH.

(*Const.*) Make (VI. 12) AB to CD as EF to PR, and (VI. 18) upon PR describe the rectilineal figure SR similar and similarly situated to either of the figures MF or NH. (*Dem.*) Then because as AB to CD, so is EF to PR, and that upon

AB and CD are described the similar and similarly sit rectilineals KAB and LCD, and upon EF and PR, in manner, the similar rectilineals MF and SR; KAB is to as MF to SR; but by the hypothesis, KAB is to LCD a to NH; and therefore the rectilineal MF having the ratio to each of the two NH and SR, these are equal t another (V. 9); they are also similar, and similarly situ therefore GH is equal to PR; and because as AB t so is EF to PR, and that PR is equal to GH; AB is t as EF to GH.

COR.—If four straight lines be proportional, their squar proportional, and conversely.

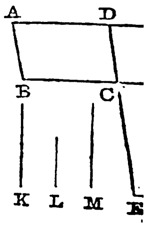
For squares are similar figures.

PROPOSITION XXIII. THEOREM.

Equiangular parallelograms have to one another the which is compounded of the ratios of their sides.

Given AC and CF two equiangular parallelograms, h the angle BCD equal to the angle ECG; *to prove that* ratio of the parallelogram AC to the parallelogram CF i same with the ratio which is compounded of the ratios of sides.

(*Const.*) Let BC and CG be placed in a straight line; the DC and CE are also in a straight line (I. 14); and con the parallelogram DG; and, taking any straight li make as BC to CG, so K to L (VI. 12); and as CE, so make L to M; (*Dem.*) therefore the ratios of K to L, and L to M, are the same with the ratios of the sides; namely, of BC to CG, and DC to CE. But the ratio of K to M is that which is said to be compounded of the ratios of K to L, and L to M (V. Def. 17); wherefore also K has to M the ratio compounded of the ratios of the sides; and because as BC to CG, so is the parallelogram AC to the parallelogram CH (VI. 1); but as BC to CG, so is K to L; therefore K is to L as the parallelogram AC to the parallelogram CH (V. 11); again, because a to CE, so is the parallelogram CH to the parallelogram CF but as DC to CE, so is L to M; wherefore L is to M parallelogram CH to the parallelogram CF; therefore, it has been proved that as K is to L, so is the parallelogra to the parallelogram CH; and as L to M, so the parallel



CH to the parallelogram CF; K is to M as the parallelogram AC to the parallelogram CF (V. 22); but K has to M the ratio which is compounded of the ratios of the sides; therefore also the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the sides.

COR. 1.—**If the terms of two analogies are lines, the rectangles under their corresponding terms are proportional.**

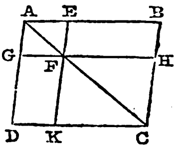
For let $A : B = C : D$, and $E : F = G : H$, and let M, N, P, Q denote respectively the rectangles $A \cdot E$, $B \cdot F$, $C \cdot G$, and $D \cdot H$; then $M : N$ in a ratio compounded of $A : B$ and $E : F$ (VI. 23), and $P : Q$ in a ratio compounded of $C : D$ and $G : H$; but the two former simple ratios are respectively equal to the two latter, and therefore the ratios compounded of them are equal (V. E); therefore $M : N = P : Q$.

COR. 2.—**Hence rectangles, whose bases are proportional, and also their altitudes, are themselves proportional.**

PROPOSITION XXIV. THEOREM.

The parallelograms about the diameter of any parallelogram are similar to the whole, and to one another.

Given a parallelogram ABCD, of which the diameter is AC; and EG and HK the parallelograms about the diameter; *to prove that* the parallelograms EG and HK are similar both to the whole parallelogram ABCD, and to one another.



(*Dem.*) Because DC and GF are parallels, the angle ADC is equal to the angle AGF (I. 29); for the same reason, because BC and EF are parallels, the angle ABC is equal to the angle AEF; and each of the angles BCD and EFG is equal to the opposite angle DAB (I. 34), and therefore they are equal to one another, wherefore the parallelograms ABCD and AEFG are equiangular; and because the angle ABC is equal to the angle AEF, and the angle BAC common to the two triangles BAC and EAF, they are equiangular to one another; therefore (VI. 4) as AB to BC, so is AE to EF; and because the opposite sides of parallelograms are equal to one another, AB is to AD as AE to AG (V. 7); and DC to CB as GF to FE; and also CD to DA as FG to GA; therefore the sides of the parallelograms ABCD and AEFG about the equal angles are proportionals; and they are therefore similar to one another (VI. Def. 9); for a like reason the parallelogram ABCD is similar to the parallelogram FHCK; wherefore each of the parallelograms GE and KH is similar to

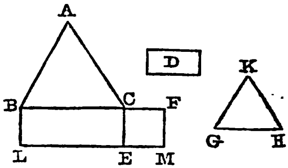
DB; but rectilineal figures which are similar to the same rectilineal figure are also similar to one another (VI. 21); therefore the parallelogram GE is similar to KH.

PROPOSITION XXV. PROBLEM.

To describe a rectilineal figure which shall be similar to one, and equal to another given rectilineal figure.

Let ABC be the *given* rectilineal figure, to which the figure to be described is required to be similar, and D that to which it must be equal. *It is required* to describe a rectilineal figure similar to ABC, and equal to D.

(*Const.*) Upon the straight line BC describe the parallelogram BE equal to the figure ABC (I. 45, Cor.); also upon CE describe the parallelogram CM equal to D, and having the angle FCE equal to the angle CBL; therefore BC and CF are in a straight line (I. 14), as also LE and EM (I. 29); between BC and CF find a mean proportional GH (VI. 13), and upon GH describe the rectilineal figure KGH similar and similarly situated to the figure ABC (VI. 18). (*Dem.*) And because BC is to GH as GH to CF, and if three straight lines be proportionals, as the first is to the third, so is the figure upon the first to the similar and similarly described figure upon the second (VI. 20, Cor. 2); therefore as BC to CF, so is the rectilineal figure ABC to KGH. But as BC to CF, so is the parallelogram BE to the parallelogram EF (VI. 1); therefore as the rectilineal figure ABC is to KGH, so is the parallelogram BE to the parallelogram EF (V. 11). And the rectilineal figure ABC is equal to the parallelogram BE; therefore the rectilineal figure KGH is equal to the parallelogram EF (V. 14). But EF is equal to the figure D (by Const.); wherefore also KGH is equal to D; and it is similar to ABC. Therefore the rectilineal figure KGH has been described similar to the figure ABC, and equal to D.

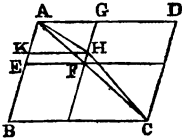


PROPOSITION XXVI. THEOREM.

If two similar parallelograms have a common angle, and be similarly situated, they are about the same diameter.

Given that the parallelograms ABCD and AEFG are similar and similarly situated, and have the angle DAB common; *to prove that* ABCD and AEFG are about the same diameter.

(*Const.*) For if not, let, if possible, the parallelogram BD have its diameter AHC in a different straight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC.



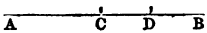
(*Dem.*) Therefore the parallelograms ABCD and AKHG being about the same diameter, they are similar to one another (VI. 24); wherefore as DA to AB, so is (VI. Def. 9) GA to AK. But because ABCD and AEFG are similar parallelograms, DA is to AB as GA to AE; therefore (V. 11) as GA to AE, so is GA to AK; wherefore GA has the same ratio to each of the straight lines AE and AK; and consequently (V. 9) AK is equal to AE, the less to the greater, which is impossible; therefore ABCD and AKHG are not about the same diameter; wherefore ABCD and AEFG must be about the same diameter.

PROPOSITION XXVII. THEOREM.

Of all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.

Let AB be a *given* straight line, which is bisected in C; and let D be any point in it; *to prove that* the square on AC is greater than the rectangle AD · DB.

(*Dem.*) For since the straight line AB is divided into two equal parts in C, and into two unequal parts in D, the rectangle contained by AD and DB, together with the square on CD, is equal to the square on AC (II. 5). The square on AC is therefore greater than the rectangle AD · DB, by the square on CD.



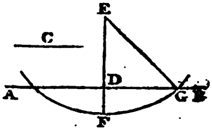
PROPOSITION XXVIII. PROBLEM.

To divide a given straight line, so that the rectangle contained by its segments may be equal to a given space; but that space must not be greater than the square on half the given line.

Let AB be the *given* straight line, and let the square upon the *given* straight line C be the space to which the rectangle contained by the segments of AB must be equal, and this square, by the determination, is not greater than that upon half the straight line AB.

(*Const.*) Bisect AB in D, and if the square upon AD be equal to the square upon C, the thing required is done. But

if it be not equal to it, AD must be greater than C, according to the determination. Draw DE at right angles to AB, and make it equal to C; produce ED to F, so that EF be equal to AD or DB, and from the centre E, at the distance EF, describe a circle meeting AB in G. Join EG.



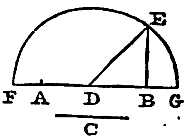
(*Dem.*) And because AB is divided equally in D, and unequally in G, the rectangle AG · GB, together with the square on DG, is equal to the square on DB (II. 5); that is, on EF or EG; but the squares on ED and DG are also equal to the square on EG (I. 47); therefore the rectangle AG · GB, together with the square on DG, is equal to the squares on ED and DG. Take away the square on DG from each of these equals; therefore the remaining rectangle AG · GB is equal to the square on ED; that is, on C. Wherefore the given line AB is divided in G, so that the rectangle contained by the segments AG and GB is equal to the square upon the given straight line C.

PROPOSITION XXIX. PROBLEM.

To produce a given straight line, so that the rectangle contained by the external segments of the given line may be equal to a given space.

Let AB be the *given* straight line, and let the square upon the given straight line C be the space to which the rectangle under the segments of AB produced, must be equal.

(*Const.*) Bisect AB in D, and draw BE at right angles to it, so that BE be equal to C; and having joined DE, from the centre D at the distance DE describe a circle meeting AB produced in G. (*Dem.*) And because AB is bisected in D, and produced to G, the rectangle AG · GB, together with the square of DB, is equal to the square on DG (II. 6), or on DE; that is, to the squares on EB and BD (I. 47). From each of these equals take the square on DB; therefore the remaining rectangle AG · GB is equal to the square on BE; that is, to the square upon C. Wherefore the line AB is produced to G, so that the rectangle contained by the segments AG and GB, of the line produced, is equal to the square on C.

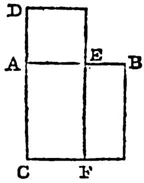


PROPOSITION XXX. PROBLEM.

To cut a given straight line in extreme and mean ratio.

Let AB be the *given* straight line; *it is required* to cut it in extreme and mean ratio.

Upon AB describe the square BC (I. 46), and produce CA to D, so that the rectangle CD · DA may be equal to the square CB (VI. 29). Take AE equal to AD, and complete the rectangle DF under DC and AE, or under DC and DA. Then because the rectangle CD · DA is equal to the square CB, the rectangle DF is equal to CB. Take away the common part CE from each, and the remainder FB is equal to the remainder DE.

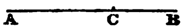


But FB is the rectangle contained by FE and EB; that is, by AB and BE; and DE is the square upon AE; therefore AE is a mean proportional between AB and BE (VI. 17); or AB is to AE as AE to EB.

But AB is greater than AE; wherefore AE is greater than EB (V. 14); therefore the straight line AB is cut in extreme and mean ratio in E (VI. Def. 4).

Otherwise, let AB be the given straight line; it is required to cut it in extreme and mean ratio.

(*Const.*) Divide AB in the point C, so that the rectangle contained by AB and BC be equal to the square on AC (II. 11). (*Dem.*) Then because the rectangle AB · BC is equal to the square on AC, as BA to AC, so is AC to CB (VI. 17); therefore AB is cut in extreme and mean ratio in C.



COR.—The section of a line in extreme and mean ratio is the same as medial section (II. Def. 6).

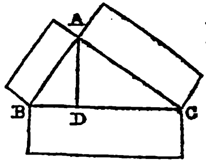
PROPOSITION XXXI. THEOREM.

In right-angled triangles, the rectilineal figure described upon the side opposite to the right angle is equal to the sum of the similar and similarly described figures upon the sides containing the right angle.

Given a right-angled triangle, ABC, having the right angle BAC; *to prove that* the rectilineal figure described upon BC is equal to the sum of the similar and similarly described figures upon BA and AC.

(*Const.*) Draw the perpendicular AD. (*Dem.*) Therefore

because in the right-angled triangle ABC, AD is drawn from the right angle at A perpendicular to the base BC, the triangles ABD and ADC are similar to the whole triangle ABC, and to one another (VI. 8); and because the triangle ABC is similar to ADB as CB to BA, so is BA to BD (VI. 4); and because these three straight lines are proportionals, as the first to the third, so is the figure upon the first to the similar and similarly described figure upon the second (VI. 20, Cor. 2). Therefore as CB to BD, so is the figure upon CB to the similar and similarly described figure upon BA; and inversely (V. A), as DB to BC, so is the figure upon BA to that upon BC; for the same reason, as DC to CB, so is the figure upon CA to that upon CB. Wherefore as BD and DC together are to BC, so are the figures upon BA and AC to that upon BC (V. 24); but BD and DC together are equal to BC; therefore the figure described on BC is equal to the similar and similarly described figures on BA and AC (V. B).

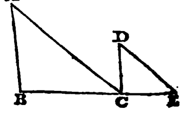


PROPOSITION XXXII. THEOREM.

If two triangles, which have two sides of the one proportional to two sides of the other, can be joined at one angle, so as to have their homologous sides parallel to one another, the remaining sides shall be in a straight line.

Given two triangles ABC and DCE, which have the two sides BA and AC proportional to the two CD and DE; namely, BA to AC as CD to DE; and let AB be parallel to DC, and AC to DE; *to prove that* BC and CE are in a straight line.

(*Dem.*) Because AB is parallel to DC, and the straight line AC meets them, the alternate angles BAC and ACD are equal (I. 29); for the same reason, the angle CDE is equal to the angle ACD; wherefore also BAC is equal to CDE; and because the triangles ABC and DCE have one angle at A equal to one at D, and the sides about these angles proportionals; namely, BA to AC as CD to DE, the triangle ABC is equiangular to DCE (VI. 6);



therefore the angle ABC is equal to the angle DCE; and the angle BAC was proved to be equal to ACD; therefore the whole angle ACE is equal to the two angles ABC and BAC; add the common angle ACB, then the angles ACE and ACB are equal to the three angles ABC, BAC, and ACB; but ABC, BAC, and ACB are equal to two right angles (I. 32); therefore

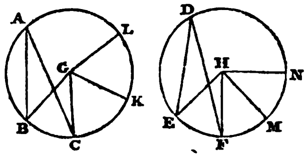
also the angles ACE, ACB are equal to two right angles; therefore (I. 14) BC and CE are in a straight line.

PROPOSITION XXXIII. THEOREM.

In equal circles, angles, whether at the centres or circumferences, have the same ratio which the arcs on which they stand have to one another; so also have the sectors.

Given two equal circles ABC and DEF; and at their centres the angles BGC and EHF, and the angles BAC and EDF at their circumferences; *to prove that* as the arc BC to the arc EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and also the sector BGC to the sector EHF.

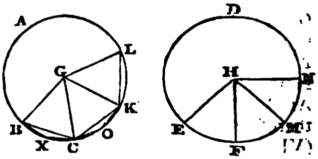
(*Const.*) Take any number of arcs CK, KL, each equal to BC, and any number whatever FM, MN, each equal to EF; and join GK, GL, HM, and HN. (*Dem.*) Because the arcs BC, CK, and KL are all equal, the angles BGC, CGK, and KGL are also all equal (III. 27); therefore what multiple soever the arc BL is of the arc BC, the same multiple is the angle BGL of the angle BGC; for the same reason, whatever multiple the arc EN is of the arc EF, the same multiple is the angle EHN of the angle EHF; and if the arc BL be equal to the arc EN, the angle BGL is also equal to the angle EHN; and if the arc BL be greater than EN, likewise the angle BGL is greater than EHN; and if less, less. There being then four magnitudes, the two arcs BC and EF, and the two angles BGC and EHF, and of the arc BC, and of the angle BGC, have been taken any equimultiples whatever; namely, the arc BL and the angle BGL; and of the arc EF, and of the angle EHF, any equimultiples whatever; namely, the arc EN, and the angle EHN; and it has been proved, that if the arc BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if less, less; therefore as the arc BC to the arc EF, so is the angle BGC to the angle EHF (V. Def. 10); but as the angle BGC is to the angle EHF, so is the angle BAC to the angle EDF (V. 15), for each is double of each (III. 20); therefore as the arc BC is to EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.



Also, as the arc BC to EF, so is the sector BGC to the sector EHF. (*Const.*) Join BC and CK, and in the arcs BC and CK, take any points X and O, and join BX, XC, CO, and OK. (*Dem.*) Then because in the triangles GBC and GCK, the

two sides BG and GC are equal to the two CG and GK, and that they contain equal angles; the base BC is equal to the base CK (I. 4), and the triangle GBC to the triangle GCK; and because the arc BC is equal to the arc CK, the remaining part of the whole circumference of the circle ABC is equal to the remaining part of the whole circumference of the same circle;

wherefore the angle BXC is equal to the angle COK; and the segment BXC is therefore similar to the segment COK (III. Def. 8); and they are upon equal straight lines BC, CK; but similar segments of circles upon equal straight lines are equal to one another (III. 24); therefore the segment BXC is equal to the segment COK; and the triangle BGC is equal to the triangle CGK; therefore the whole, the sector BGC, is equal to the whole, the sector CGK;



for the same reason, the sector KGL is equal to each of the sectors BGC, CGK; in the same manner the sectors EHF, FHM, and MHN may be proved equal to one another; therefore what multiple soever the arc BL is of the arc BC, the same multiple is the sector BGL of the sector BGC; for the same reason, whatever multiple the arc EN is of EF, the same multiple is the sector EHN of the sector EHF; and if the arc BL be equal to EN, the sector BGL is equal to the sector EHN; and if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; and if less, less; since, then, there are four magnitudes, the two arcs BC and EF,

and the two sectors BGC and EHF, and of the arc BC, and sector BGC, the arc BL and sector BGL are any equal multiples whatever; and of the arc EF, and sector EHF, the arc EN and sector EHN are any equimultiples whatever;

and that it has been proved, if the arc BL be greater than EN, the sector BGL is greater than the sector EHN; and if equal, equal; and if less, less; therefore (V. Def. 10), as the arc BC is to the arc EF, so is the sector BGC to the sector EHF.

Cor. 1.—Similar sectors of the same, or of equal circles, are equal.

Cor. 2.—An angle at the centre of a circle, is to four right angles, as the arc on which it stands, to the circumference of the circle.

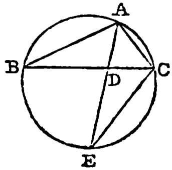
For an angle at the centre, is to one right angle, as the arc subtending the former one to a quadrantal arc; therefore (V. 4) the angle at the centre, is to four right angles, as the arc subtending it, to the whole circumference.

PROPOSITION B. THEOREM.

If an angle of a triangle be bisected by a straight line, which likewise cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line bisecting the angle.

Given a triangle ABC, and the angle BAC bisected by the straight line AD; *to prove that* the rectangle BA · AC is equal to the rectangle BD · DC, together with the square on AD.

(*Const.*) Describe the circle ACB about the triangle (IV. 5), and produce AD to the circumference in E, and join EC.

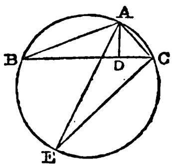


(*Dem.*) Then because the angle BAD is equal to the angle CAE, and the angle ABD to the angle AEC (III. 21), in the same segment; the triangles ABD and AEC are equiangular to one another; therefore as BA to AD, so is EA to AC (VI. 4), and consequently the rectangle BA · AC is equal to the rectangle EA · AD (VI. 16); that is, to the rectangle ED · DA, together with the square on AD (II. 3); but the rectangle ED · DA is equal to the rectangle BD · DC (III. 35); therefore the rectangle BA · AC is equal to the rectangle BD · DC, together with the square on AD.

PROPOSITION C. THEOREM.

If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Given a triangle ABC, and AD the perpendicular from the angle A to the base BC; *to prove that* the rectangle BA · AC is equal to the rectangle contained by AD and the diameter of the circle described about the triangle.



(*Const.*) Describe (IV. 5) the circle ACB about the triangle, and draw its diameter AE, and join EC. (*Dem.*) Because the right angle BDA is equal to the angle ECA in a semicircle (III. 31), and the angle ABD to the angle AEC in the same segment (III. 21); the triangles ABD and AEC are equiangular ;

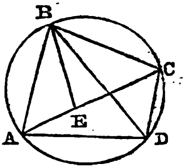
therefore BA is to AD as EA to AC (VI. 4); and consequently the rectangle BA · AC is equal to the rectangle EA · AD (VI. 16).

PROPOSITION D. THEOREM.

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

Given any quadrilateral ABCD inscribed in a circle, and join AC and BD; *to prove that* the rectangle contained by AC and BD is equal to the two rectangles contained by AB and CD, and by AD and BC.

(*Const.*) Make the angle ABE equal to the angle DBC; (*Dem.*) add to each of these the common angle EBD, then the angle ABD is equal to the angle EBC; and the angle BDA is equal to the angle BCE (III. 21), because they are in the same segment; therefore the triangle ABD is equiangular to the triangle BCE; wherefore (VI. 4) BC is to CE as BD to DA; and consequently the rectangle BC · AD is equal to the rectangle BD · CE (VI. 16); again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC (III. 21), the triangle ABE is equiangular to the triangle BCD; therefore BA is to AE as BD to DC; wherefore the rectangle BA · DC is equal to the rectangle BD · AE; but the rectangle BC · AD has been shewn equal to the rectangle BD · CE; therefore the whole rectangle AC · BD is equal to the rectangle AB · DC, together with the rectangle AD · BC (II. 1).

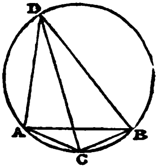


PROPOSITION E. THEOREM.

If a segment of a circle be bisected, and from the extremities of the base of the segment, and from the point of bisection, straight lines be drawn to any point in the circumference, the sum of the two lines drawn from the extremities of the base, will have to the line drawn from the point of bisection the same ratio which the chord of the segment has to the chord of half the segment.

Given a circle ABD, of which AB is a segment bisected in C, and from A, C, and B to D, any point whatever in the circumference, let AD, CD, and BD be drawn; *to prove that* the sum of the two lines AD and DB has to DC the same ratio that BA has to AC.

(*Dem.*) For since ACBD is a quadrilateral inscribed in a circle, of which the diagonals are AB and CD, the rectangles AD · CB and DB · AC are together equal to the rectangle AB · CD (VI. D). But the rectangle AD · BC is equal to the rectangle AD · AC, because BC is equal to AC; and therefore the two rectangles AD · AC and BD · AC are equal to the rectangle AB · CD.



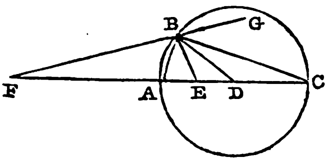
But the two rectangles AD · AC and BD · AC are the rectangle contained by AC, and the sum of the lines AD and DB (II. 1); wherefore the rectangle contained by AC and the sum of the lines AD and DB is equal to the rectangle AB · CD; and because the sides of equal rectangles are reciprocally proportional (VI. 14), the sum of AD and DB is to DC as AB to AC.

PROPOSITION F. THEOREM.

If two points be taken in the diameter of a circle, such that the rectangle contained by the segments intercepted between them and the centre of the circle be equal to the square on the semi-diameter; and if from these points two straight lines be inflected to any point whatsoever in the circumference of the circle, the ratio of the lines inflected will be the same with the ratio of the segments intercepted between the two first-mentioned points and the circumference of the circle.

Given a circle ABC, of which the centre is D, and in DA produced, let the points E and F be such that the rectangle ED · DF is equal to the square on AD; from E and F to any point B in the circumference, let EB, FB be drawn; *it is required to prove that* FB : BE :: FA : AE.

(*Const.*) Join BD and BA, (*Dem.*) and because the rectangle FD · DE is equal to the square on AD, that is, on DB; FD : DB :: DB : DE (VI. 17). The two triangles FDB and BDE have therefore the sides proportional that are about the common angle D; therefore they are equiangular (VI. 6), the angle DEB being equal to the angle DBF, and DBE to DFB.



Now, since DB is equal to DA, the angle DBA is equal to the angle DAB (I. 5); but the angle DAB is equal to the angles AFB and FBA (I. 32), and the angle DBA is equal to

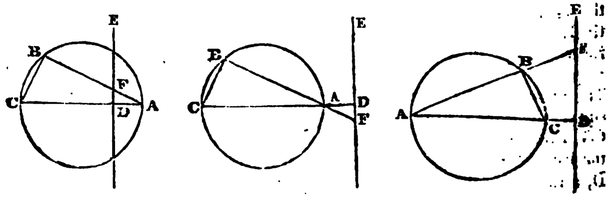
the two angles ABE and EBD, therefore the angles AFB and FBA are equal to the angles ABE and EBD; but the angle AFB is equal to EBD, therefore the remaining angle FBA is equal ABE, wherefore the line AB bisects the vertical angle FBE of the triangle FBE; therefore (VI. 3) FB : BE :: FA : AE.

COR.—If FB be produced to G, and BC joined; the two angles FBE and EBG are equal to two right angles (I. 13); but ABC, being an angle in a semicircle, is a right angle (III. 31); and the angle ABC is equal to the two angles ABE and EBC, therefore the two angles FBE and EBG are equal to twice the angles ABE and EBC; but by the proposition the angle FBE is equal to twice the angle ABE, therefore the remaining angle EBG is equal to twice the angle EBC, wherefore the exterior angle EBG is bisected by the straight line BC; therefore (VI. A) FB : BE :: FC : CE, but FB : BE :: FA : AE, therefore (V. 11) FC : CE :: FA : AE, and FC is therefore divided harmonically in A and E.

PROPOSITION G. THEOREM.

If, from one extremity of the diameter of a circle, a chord be drawn, and a perpendicular be drawn to the diameter, so as to cut it and the chord either internally or externally, the rectangle under the diameter and its segment by the perpendicular reckoned from that extremity, is equal to the rectangle under the chord and its corresponding segment.

Given a circle ABC, of which AC is a diameter, and DE a perpendicular to the diameter AC, and let AB meet DE in



F; *to prove that* the rectangle BA · AF is equal to the rectangle CA · AD.

(*Const.*) Join BC. (*Dem.*) And because ABC is an angle in a semicircle, it is a right angle (III. 31). Now, the angle ADF is also a right angle (Hyp.); and the angle BAC is either the same with DAF, or vertical to it; therefore the triangles ABC, ADF are equiangular, and BA : AC :: AD : AF

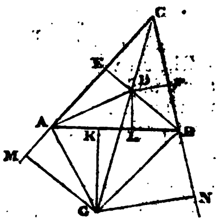
(VI. 4); therefore also the rectangle BA · AF, contained by the extremes, is equal to the rectangle AC · AD, contained by the means (VI. 16).

PROPOSITION H. THEOREM.

If the angles at the base of a triangle be bisected by two lines that meet, and the exterior angles at the base, formed by producing the two sides, be similarly bisected, the two points of concourse and the vertex shall be in one straight line, which shall bisect the vertical angle.

Given a triangle ABC, of which the angles at the base are bisected by the lines AD and BD, which meet in D, and the exterior angles are bisected by AG and BG; *to prove that* the two points D, G, and the vertex C, are in one straight line, which bisects the angle at C.

(*Const.*) For draw DE, DF, DL, and GM, GK, GN perpendicular to the sides and sides produced. (*Dem.*) Then in the triangles ADE, ADL, the angles at A are equal, and the angles at E and L are also equal, being right angles, and AD is common to them; therefore (I. 26) AL = AE and DL = DE. It is proved in a similar manner that BL = BF, and DL = DF; also that AM = AK, GM = GK; that BN = BK, and GN = GK; and since DE and DF are each equal to DL, therefore DE = DF; and similarly GM = GN. Again, since in the triangles CED, CFD, the two sides CD and DE are respectively equal to CD and DF, and the angles at E and F are right, consequently (I. C.) the triangles are equal in every respect; and therefore the angle at C is bisected by CD. The triangles CMG and CNG are similarly proved (I. C.) to be equal in every respect; hence the angle at C is bisected by CG, but it was also shewn to be bisected by CD; therefore the lines CG, CD coincide; and therefore the three points C, D, and G are in the same straight line.



PROPOSITION K. THEOREM.

In reference to the triangle of proposition H, the segments of each side produced that are intercepted between the vertex and the external perpendiculars, are each equal to the semiperimeter of the triangle, the segments of these sides next the vertex are equal to the excess of the semiperimeter above the base, and the

segment of each of these sides next the base is respectively equal to the excess of the semiperimeter above the other side.

In the diagram of Prop. H. or L., CM or CN = S, if S denote the semiperimeter; CE or CF = S − AB, AE = S − BC, and BF = S − AC.

(*Dem.*) Since AK = AM, and BK = BN, therefore CM + CN = perimeter, and CM = CN = S, the semiperimeter; and AM = S − AC, and BN = S − BC. Also CE + AE + AM = semiperimeter = CF + FB + BN, and CE = CF, therefore AE and AM are equal to FB and BN, or their equals AL + AK = BK + BL, that is 2AK + KL = 2BL + KL, take KL from both, then 2AK = 2BL or AK = BL; to each of these equals add KL, then AL = BK, but AL = AE, therefore BK = AE, and AK = AM, hence BK + KA = EA + AM, or AB = EM; therefore CE or CM − EM = S − AB.

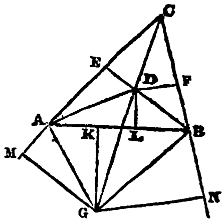
Also BN = BK = AL = AE = S − BC. Wherefore CM, CE, EA, and AM are respectively equal to S, S − AB, S − BC, and S − AC.

PROPOSITION L. THEOREM.

The area of a triangle is a mean proportional between the rectangle under the semiperimeter and its excess above the base, and that under its respective excesses above the other two sides.

It is required to prove that the triangle ABC is a mean proportional between the rectangles CM · CE, and AM · AE, or between S(S − AB) and (S − AC) (S − BC).

(*Dem.*) For since the angle CED is equal to the angle CMG, each being a right angle, and the angle MCG is common to the two triangles CED and CMG, they are similar, therefore (VI. 4) CE : ED = CM : MG; but CM : CM = ED : ED, therefore CE · CM : ED · CM = CM · ED : MG · ED (VI. 23, Cor. 1).



But the triangle ABC is equal to the three ADC, ADB, and CDB, which are respectively equal to half the rectangles AC · DE, AB · DL, BC · DF (I. 41), the sum of which is = ED · CM.

And since angles EAL + MAK = two right angles, EAD + MAG = one right angle = AGM + MAG (I. 32), and therefore EAD = AGM, and the angles at E and M are right angles, hence the triangles AGM and EAD are similar, and AE : ED = MG : AM; therefore AE · AM = MG · ED. The former analogy therefore becomes CM · CE

s ABC = ABC : AM · AE, or S(S − AB) : ABC = ABC : (S − AC) (S − BC).

COR.—If the sides opposite to the angles A, B, C, be respectively denoted by a, b, and c, then

$$s(s-c) : ABC = ABC : (s-a)(s-b).$$

Therefore $$(ABC)^2 = s(s-a)(s-b)(s-c);$$

or area of $$ABC = \sqrt{s(s-a)(s-b)(s-c)}.$$

Which is the useful rule by which the area of a triangle is computed, when its three sides are given.

EXERCISES.

1. Lines that meet any three parallel lines, are cut proportionally.

2. If a straight line be divided into three segments such, that the rectangle under the whole line and the middle segment is equal to that under the extreme segments, the line is cut harmonically.

3. If from any point in the circumference of a circle a perpendicular be drawn on any radius, and a tangent from the same point to meet the radius produced, the radius will be a mean proportional between its segments, intercepted between the centre and the points of concourse.

4. If arcs of different circles have a common chord, lines diverging from one of its extremities will cut the arcs proportionally.

5. To cut a given line harmonically.

6. To find a line such, that the first of two given lines shall be to the second as the square of the first to the square of the required line.

7. To find a line such, that the first of two given lines shall be to it as the square of the first to the square of the second.

8. From one angle of a triangle a line is drawn to the point of bisection of the opposite side, and through its middle point another is drawn from either angle to the side subtending it; prove that this is divided in the ratio of 2 to 1.

9. If a square be inscribed in a right-angled triangle, one side being on the hypotenuse, the hypotenuse is divided in continued proportion.

10. The part of a tangent to a circle, intercepted between two tangents drawn from the extremities of any diameter, subtends a right angle at the centre; and is divided at the point of contact, so that the radius is a mean proportional between the segments.

11. If through the bisection of the base of a triangle any line be drawn, cutting one side of the triangle, and the other side produced, and a line be drawn through the vertex parallel to the base, the former line will be cut harmonically.

12. If one side of a right-angled triangle be double of the other, prove that a perpendicular from the right angle on the hypotenuse will divide it into segments in the ratio of 4 to 1.

13. From a given angle, to cut off a triangle equal to a given space.

14. To cut off from a given triangle another similar to it, and in a given ratio to it.

15. Given the base, the altitude, and the ratio of the sides of a triangle, to construct it.

16. If from the extremities of the base of a triangle, lines be drawn bisecting the opposite sides, they will divide each other in the ratio of 2 to 1.

17. If a line be drawn parallel to the base of a triangle to meet the sides, and the alternate extremities of this line and of the base be joined, the line drawn from the vertex through the intersection of the connecting lines will bisect the base, and will be cut harmonically.

18. The inclination of two chords of a circle is measured by half the sum, or half the difference of the intercepted arcs, according as they intersect internally or externally.

This property of the circle suggested a considerable improvement in the form of astronomical angular instruments.

19. If three lines be in continued proportion, the first is to the third as the square on the difference between the first and second, to the square on the difference between the second and third.

20. If a line bisect the angle adjacent to the vertical angle of a triangle, and meet the base produced, the difference between the square on that line and the rectangle under the external segments of the base, is equal to the rectangle under the sides of the triangle.

This may be considered as another case of VI. B.

21. If two tangents and a secant be drawn to a circle from a

point without it, and the points of contact be joined by a straight line, the secant will be cut harmonically.

22. If two triangles have two angles together equal to two right angles, and other two angles equal, the sides about their remaining angles are proportional.

23. If a line be drawn through any point in the base of an isosceles triangle, so as to cut off from one side and add to the other equal segments, it will be bisected by the base.

24. If from the angular points of a triangle, lines be drawn through any point within it to meet the opposite sides, and if from the point of section of the base, lines be drawn through the other two points of section to meet a line drawn through the vertex parallel to the base, the intercepted portion of the latter is bisected in the vertex.

25. If from the extremities of the base of a triangle, lines be drawn through any point in the perpendicular to meet the sides, lines joining the points of section of the sides with that of the base, will make equal angles with the base.

26. Harmonicals cut all straight lines that intersect them harmonically.

27. If lines be drawn from the angular points of a triangle, through any point within it, to meet the opposite sides, and the points of section be joined, the former lines will be cut harmonically; and if these lines be produced to meet the sides produced, the latter will be cut harmonically.

28. If through any point within or without a triangle, lines be drawn from the angular points to meet the opposite sides, and if lines joining the points of section be produced to meet the sides produced if necessary, the three points of concourse are in one line.

29. To draw a straight line, so that the part of it intercepted between one side of a given isosceles triangle and the other side produced, shall be equal to a given line, and be bisected by the base.

30. Given the altitude, the vertical angle, and the sum or difference of the sides of a triangle, to construct it.

31. Given the base, the altitude, and the sum or difference of the sides of a triangle, to construct it.

32. Given the altitude of a triangle, the difference of the angles at the base, and the sum or difference of the sides, to construct it.

33. The altitude, the difference of the segments of the base, and either the sum or difference of the two sides of a triangle, are given to construct it.

34. Given the altitude, the vertical angle, and the perimeter of a triangle, to construct it.

35. If perpendiculars be drawn from the extremities of the base of a triangle on a straight line that bisects the angle opposite to the base, the area of the triangle is equal to the rectangle contained by either of the perpendiculars, and the segment of the bisecting line between the angle and the other perpendicular.

36. If perpendiculars be drawn from the extremities of the base of a triangle on a straight line which bisects the angle opposite to the base; four times the rectangle contained by the perpendiculars is equal to the rectangle contained by two straight lines, one of which is the base increased by the difference of the sides, and the other the base diminished by the difference of the sides.

37. If perpendiculars be drawn from the extremities of the base of a triangle on a straight line which bisects the angle opposite to the base; four times the rectangle contained by the segments of the bisecting line between the angle and the perpendiculars, is equal to the rectangle contained by two straight lines, one of which is the sum of the sides increased by the base, and the other the sum of the sides diminished by the base.

38. Apply the three exercises (35, 36, and 37) to prove proposition L of the Sixth Book.

ON THE QUADRATURE OF THE CIRCLE, AND THE RECTIFICATION OF ITS CIRCUMFERENCE.

DEFINITIONS.

1. The determination of a square equal to a given surface, is called the *quadrature* of that surface.

2. The determination of a straight line equal to a curve line, is called the *rectification* of that curve line.

3. A *mixed line* is composed of straight and curve lines.

4. A *mixtilineal* space is a space contained by a mixed line.

5. The *inclination* of a straight line and a curve is the angle contained by the former, and a tangent to the latter at the point of intersection.

6. The *supplemental* chord of an arc is the chord of its defect from a semicircumference.

AXIOMS.

1. Of all lines that can join two points, there must be at least one such that no other is less than it; and if there be only one such, it is the least.

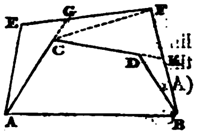
2. If any number of lines fulfil certain conditions, and if a line fulfilling the same conditions can always be found less than any of them, except one, this one must be the least.

PROPOSITION I. THEOREM.

If two rectilineal figures be on the same side of the same base, and if one of them be wholly encompassed by the other, and be also concave internally, the sum of its sides is less than that of the other.

Given the figures ACDB, AEFB upon the same base AB; to *prove that* $AE + EF + FB > AC + CD + DB$.

(*Const.*) Produce AC to G; join C, F, and produce CD to K. (*Dem.*) Then (I. 20) $AE + EG > AG$, $CG + GF > CF$, $CF + FK > CK$, and $DK + KB > DB$. Therefore $AE + EF + FB > AG + GF + FB$, which is $> AC + CF + FB$, which is $> AC + CK + KB$, which is $> AC + CD + DB$, still more then is $AE + EF + FB > AC + CD + DB$.



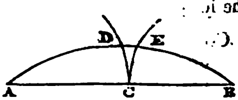
And in a similar manner the proposition is proved for polygons.

PROPOSITION II. THEOREM.

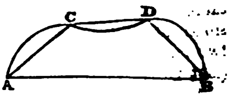
Of all the lines, straight and curve, that can join two points, the straight line is the least.

Let A, B, be any two points, the straight line AB is the shortest line that can join them.

1. Let the curve line ADB, concave towards AB, join them. Then take any point C in AB, and from the centres A and B, with the radii AC and CB respectively, describe the arcs CD and CE, passing through C, and cutting ADB in D and E. (*Dem.*) Then, since the arcs touch at C (III. 12), and cannot touch at any other point (III. 13), the point E must be without the arc CD, and D without the arc CE; and the line DE between them must be of some length. Now, if the points D and E be made to coincide with C, A and B remaining fixed, the curve lines AD and EB would connect A and B; and hence a line shorter than ADEB can connect these points. The same can be proved of any other concave line, therefore the straight line AB is shorter than any of them (Ax. 1).



2. Let a line ACDB, partly concave and partly convex towards AB, connect the points. Let AC and DB be concave, and CD the convex portion. Draw the straight lines AC, CD, and DB. Then (by 1st case) the straight lines AC, CD, and DB are less than the curve lines AC, CD, and DB respectively; and the sum of the former is therefore less than the whole curve line; hence a line shorter than it has been found.

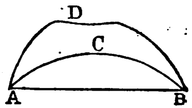


3. Let the crooked line ACDB join A and B. Then, if AD be joined by a straight line, $AC + CD > AD$, and $AD + DB$

$>$ AB; therefore the sum of the straight lines AC, CD, and DB are greater than AB.

Since, therefore, a line shorter than any of the curve or crooked lines that join A and B can be found, but none shorter than the straight line can be found, therefore it is the shortest (Ax. 1).

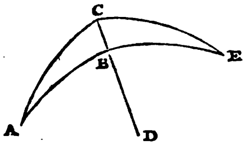
COR. 1.—If two points A and B be joined by a straight line AB, and a curve line ACB, or a mixed line, which is concave towards AB; ACB is the least of all the lines that lie on the same side of AB, that join A and B, and that do not lie between ACB and AB.



For if ADB be any other line, then by joining two points in it by a straight line, a shorter line will be found than it, joining A and B, and fulfilling the other conditions; and this will be the case whether ADB be composed of straight lines, or be a mixed line, or a curve line. And as this can be proved of every line fulfilling the required conditions, except of ACB, therefore (Ax. 2) ACB is the least.

COR. 2.—If two intersecting curves, whose curvatures lie in one and the same direction, be cut by a straight line, inclined to the interior at an angle not greater than a right angle, the arc which it intercepts on this line from the point of intersection will be less than the corresponding arc of the other line.

Let AB and AC intersect in A, and be cut by CD, so that the angle at B is not greater than a right angle, then AC $>$ AB.



For if CE, BE be curves equal to AC, AB in every respect, on the other side of CD, but in a reverse position, and having respectively the same inclinations to CD; then since the angles at B do not exceed two right angles, ABE is concave towards D, and hence (last Cor.) ACE $>$ ABE, and therefore AC $>$ AB.

COR. 3.—Hence the circumference of a circle is greater than the perimeter of any inscribed polygon, and less than that of any one circumscribing it.

PROPOSITION III. THEOREM.

If from the greater of two unequal magnitudes there be t away its half, and from the remainder its half, and so on, shall at length remain a magnitude less than the least o proposed magnitudes.

Let AB and C be two unequal magnitudes, of which AB i greater. If from AB there be taken away its half, and the remainder its half, and so on, there shall at length re a magnitude less than C.

(*Const.*) For C may be multiplied so as at length to be greater than AB. Let it be so multiplied, and let D multiple be greater than AB, and let DE be divided into DF, FG, and GE, each equal to C. From AB take BH equal to its half, and from the remainder AH take HK equal to its half, and so on, until there be as many divisions in AB as there are in DE; and let the divisions in AB be AK, KH, and HB; and the divisions in ED be DF, FG, and GE. (*Dem.*) And because DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is equal to its half; therefore the remainder GD is greater than the remainder HA. Again, because GD is greater than HA, and that GF is not greater than the half of GD, but HK is equal to the half of HA; therefore the remainder FD is greater the remainder AK; and FD is equal to C; therefore greater than AK; that is, AK is less than C.

PROPOSITION IV. THEOREM.

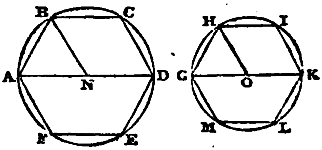
Equilateral polygons of the same number of sides inscrib circles are similar, and are to one another as the squares o diameters of the circles.

Given ABCDEF and GHIKLM two equilateral polygons o same number of sides inscribed in the circles ABD and GHI *prove that* ABCDEF and GHIKLM are similar, and a one another as the squares on the diameters of the circles A GHK.

(*Const.*) Find N and O the centres of the circles; joi and BN, as also GO and HO, and produce AN and G they meet the circumferences in D and K.

(*Dem.*) Because the chords AB, BC, CD, DE, EF, and F all equal, the arcs AB, BC, CD, DE, EF, and FA are

equal (III. 28); for the same reason, the arcs GH, HI, IK, KL, LM, and MG are all equal, and they are equal in number to the others; therefore whatever part the arc AB is of the whole circumference ABD, the same part is the arc GH of the circumference GHK; but the angle ANB is the same part of four right angles that the arc AB is of the circumference ABD (VI. 33, Cor. 2); and the angle GOH is the same part of four right angles that the arc GH is of the circumference GHK; therefore the angles ANB and GOH are each of them the same part of four right angles, and therefore they are equal to one another.



The isosceles triangles ANB and GOH are therefore equiangular (VI. 6), and the angle ABN equal to the angle GHO; in the same manner, by joining NC, OI, it may be proved that the angles NBC and OHI are equal to one another, and to the angle ABN; therefore the whole angle ABC is equal to the whole angle GHI; and the same may be proved of the angles BCD and HIK, and of the rest; therefore the polygons ABCDEF and GHIKLM are equiangular to one another; and since they are equilateral, the sides about the equal angles are proportionals; the polygon ABCDEF is therefore similar to the polygon GHIKLM; and because similar polygons are as the squares on their homologous sides (VI. 20), the polygon ABCDEF is to the polygon GHIKLM as the square on AB to the square on GH; but because the triangles ANB and GOH are equiangular, the square on AB is to the square on GH as the square on AN to the square on GO (VI. 4, and 22, Cor.), or as four times the square on AN to four times the square on GO; that is, as the square on AD to the square on GK; therefore also the polygon ABCDEF is to the polygon GHIKLM as the square on AD to the square on GK; and they have also been shewn to be similar.

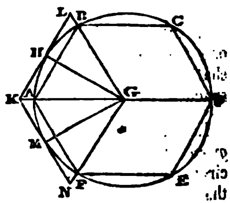
COR.—Every equilateral polygon inscribed in a circle is also equiangular. For the isosceles triangles, which have their common vertex in the centre, are all equal and similar; therefore the angles at their bases are all equal, and the angles of the polygon, which are the doubles of these angles, are therefore also equal.

PROPOSITION V. PROBLEM.

The side of any equilateral polygon inscribed in a circle being given, to find the side of a polygon of the same number of sides described about the circle.

Given ABCDEF an equilateral polygon inscribed in the circle ABD; *it is required* to find the side of an equilateral polygon of the same number of sides described about the circle.

(*Const.*) Find G the centre of the circle; join GA, GB; bisect the arc AB in H; and through H draw KHL, touching the circle in H, and meeting GA and GB produced in K and L; KL is the side of the polygon required.



Produce GF to N, so that GN may be equal to GL; join KN, and from G draw GM at right angles to KN; join also HG.

(*Dem.*) Because the arc AB is bisected in H, the angle AGH is equal to the angle BGH, and because KL touches the circle in H, the angles LHG and KHG are right angles; therefore there are two angles of the triangle HGK equal to two angles of the triangle HGL, each to each; but the side GH is common to these triangles; therefore they are equal, and GL is equal to GK. Again, in the triangles KGL and KGN, because GN is equal to GL, and GK common, and also the angle LGK equal to the angle KGN; therefore the base KL is equal to the base KN and the angle GKL to GKN; but because the triangle KGN is isosceles, the angle GKN is equal to the angle GNK, and the angles GMK and GMN are both right angles by construction; wherefore the triangles GMK and GMN have two angles of the one equal to two angles of the other, and they have also the side GM common; therefore they are equal, and the side KM is equal to the side MN, so that KN is bisected in M; but KN is equal to KL, and therefore their halves KM and KH are also equal; wherefore in the triangles GKH and GKM, the two sides GK and KH are equal to the two sides GK and KM, each to each; and the angles GKH, GKM are also equal, therefore GM is equal to GH; wherefore the point M is in the circumference of the circle; and because KMG is a right angle, KM touches the circle; and in the same manner, by joining the centre and the other angular points of the inscribed polygon, an equilateral polygon may be described about the circle, the sides of which will each be equal to KL, and will be equal in number to the sides of the inscribed polygon; therefore KL is the side of an equilateral polygon described about the circle of the same number of sides with the inscribed polygon ABCDEF, which was to be found.

COR.—Because GL, GK, and GN, and the other straight lines drawn from the centre G to the angular points of the polygon

described about the circle ABD, are all equal; if a circle be described from the centre G, with the distance GK, the polygon will be inscribed in that circle; and therefore it is similar to the polygon ABCDEF (4).

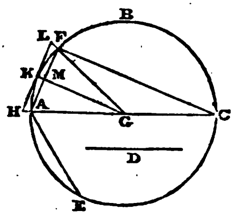
PROPOSITION VI. THEOREM.

A circle being given, two similar polygons may be found, the one described about the circle, and the other inscribed in it, which shall differ from one another by a space less than any given space.

Let ABC be the *given* circle, and the square on D any *given* space; *to prove that* a polygon may be inscribed in the circle ABC, and a similar polygon described about it, so that the difference between them shall be less than the square on D.

(*Const.*) In the circle ABC apply the straight line AE equal to D; and let AB be a fourth part of the circumference of the circle. From the circumference AB take away its half, and from the remainder its half, and so on, till the circumference AF is found less than the circumference AE. Find the centre G; draw the diameter AC, as also the straight lines AF and FG; and having bisected the circumference AF in K, join KG, and draw HL, touching the circle in K, and meeting GA and GF produced in H and L; join CF.

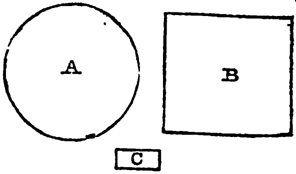
(*Dem.*) Because the isosceles triangles HGL and AGF have the common angle AGF, they are equiangular (VI. 6), and the angles GHK and GAF are therefore equal to one another; but the angles GKH and CFA are also equal, for they are right angles; therefore the triangles HGK and ACF are likewise equiangular.



And because the arc AF was found by taking from the arc AB its half, and from that remainder its half, and so on, AF will be contained a certain number of times exactly in the arc AB, and therefore it will also be contained a certain number of times exactly in the whole circumference ABC; and the straight line AF is therefore the side of an equilateral polygon inscribed in the circle ABC; wherefore also HL is the side of an equilateral polygon of the same number of sides described about ABC (5). Let the polygon described about the circle be called M, and the polygon inscribed be called N; *then* because these polygons are similar (5, Cor.),

they are as the squares on the homologous sides HL and AF (VI. 20, Cor. 3), that is, because the triangles HLG and AFG are similar, as the square on HG to the square on AG, that is, on GK; but the triangles HGK and ACF have been proved to be similar, and therefore the square on AC is to the square on CF as the polygon M to the polygon N; and by conversion, the square on AC is to its excess above the square on CF; that is, to the square on AF as the polygon M to its excess above the polygon N; but the square on AC, that is, the square described about the circle ABC, is greater than the equilateral polygon of eight sides described about the circle, because it contains that polygon; and, for the same reason, the polygon of eight sides is greater than the polygon of sixteen, and so on; therefore the square on AC is greater than any polygon described about the circle by the continual bisection of the arc AB; it is therefore greater than the polygon M. Now it has been demonstrated that the square on AC is to the square on AF as the polygon M to the difference of the polygons; therefore since the square on AC is greater than M, the square on AF is greater than the difference of the polygons. The difference of the polygons is therefore less than the square on AF; but AF is less than D; still more then is the difference of the polygons less than the square on D; that is, than the given space.

COR. 1.—Because the polygons M and N differ from one another more than either of them differs from the circle, the difference between each of them and the circle is less than the given space—namely, the square of D; and, therefore, however small any given space may be, a polygon may be inscribed in the circle, and another described about it, each of which shall differ from the circle by a space less than the given space.



COR. 2.—The space B, which is greater than any polygon that can be inscribed in the circle A, and less than any polygon that can be described about it, is equal to the circle A. If not, let them be unequal; and first let B exceed A by the space C; then, because the polygons described about the circle A are all greater than B, by hypothesis, and because B is greater than A by the space C, therefore no polygon can be described about the circle A, but what must exceed it by a space greater than C, which is absurd. In the same manner, if B be less than A by the space C, it is shewn

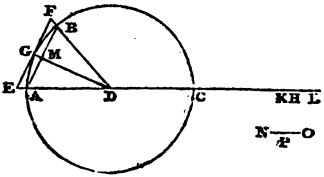
that no polygon can be inscribed in the circle A, but is less than A by a space greater than C, which is also absurd; therefore A and B are not unequal; that is, they are equal to one another.

PROPOSITION VII. THEOREM.

Every circle is equal to the rectangle contained by the semi-diameter, and a straight line equal to half the circumference.

Let ABC be a circle of which the centre is D, and the diameter AC; if in AC produced there be taken AH equal to half the circumference; it is required *to prove that* the circle is equal to the rectangle contained by DA and AH.

(*Const.*) Let AB be the side of any equilateral polygon inscribed in the circle ABC; bisect the circumference AB in G, and through G draw EGF, touching the circle, and meeting DA produced in E, and DB produced in F; EF will be the side of an equilateral polygon described about the circle ABC (5).



In AC produced take AK equal to half the perimeter of the polygon whose side is AB, and AL equal to half the perimeter of the polygon whose side is EF; then AK will be less, and AL greater than the straight line AH (2, Cor. 3).

(*Dem.*) Now because in the triangle EDF, DG is drawn perpendicular to the base, the triangle EDF is equal to the rectangle contained by DG and the half of EF; and as the same is true of all the other equal triangles having their vertices in D, which make up the polygon described about the circle; therefore the whole polygon is equal to the rectangle contained by DG and AL, half the perimeter of the polygon, or by DA and AL; but AL is greater than AH, therefore the rectangle DA · AL is greater than the rectangle DA · AH; the rectangle DA · AH is therefore less than the rectangle DA · AL; that is, than any polygon described about the circle ABC.

Again, the triangle ADB is equal to the rectangle contained by DM the perpendicular and one-half of the base AB, and it is therefore less than the rectangle contained by DG or DA, and the half of AB; and as the same is true of all the other triangles having their vertices in D, which make up the inscribed polygon, therefore the whole of the inscribed polygon is less than the rectangle contained by DA, and AK half the perimeter of the polygon. Now the rectangle DA · AK is less than DA · AH; therefore *the* polygon whose side is AB is still less than

DA · AH; and the rectangle DA · AH is therefore greater than any polygon inscribed in the circle ABC; but the same rectangle DA · AH has been proved to be less than any polygon described about the circle ABC; therefore the rectangle DA · AH is equal to the circle ABC (6, Cor. 2). Now DA is the semidiameter of the circle ABC, and AH the half of its circumference.

COR. 1.—Hence a polygon may be described about a circle, the perimeter of which shall exceed the circumference of the circle by a line that is less than any given line. Let NO be the given line. Take in NO the part NP less than its half, and less also than AD, and let a polygon be described about the circle ABC, so that its excess above ABC may be less than the square of NP (6, Cor. 1). Let the side of this polygon be EF; and since, as has been proved, the circle is equal to the rectangle DA · AH, and the polygon to the rectangle DA · AL, the excess of the polygon above the circle is equal to the rectangle DA · HL, therefore the rectangle DA · HL is less than the square of NP; and therefore since DA is greater than NP, HL is less than NP, and twice HL less than twice NP, wherefore twice HL is still less than NO; but HL is the difference between half the perimeter of the polygon whose side is EF, and half the circumference of the circle; therefore twice HL is the difference between the whole perimeter of the polygon and the whole circumference of the circle. The difference, therefore, between the perimeter of the polygon and the circumference of the circle, is less than the given line NO.

COR. 2.—Hence, also, a polygon may be inscribed in a circle, such that the excess of the circumference above the perimeter of the polygon may be less than any given line. This is proved like the preceding.

COR. 3.—Hence (VI. 33), the area of a sector is equal to half the rectangle under its arc and the radius.

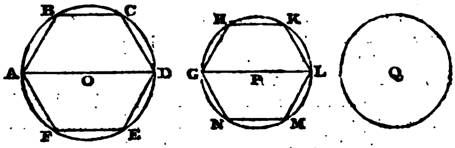
PROPOSITION VIII. THEOREM.

Circles are to one another in the duplicate ratio, or as the squares on their diameters.

Given two circles ABD and GHL, of which the diameters are AD and GL; *to prove that* the circle ABD is to the circle GHL as the square on AD to the square on GL.

(*Const.*) Let ABCDEF and GHKLMN be two equilateral polygons of the same number of sides, inscribed in the circles ABD and GHL; and let Q be such a space that the square

on AD is to the square on GL as the circle ABD to the space Q. (*Dem.*) Because the polygons ABCDEF and GHKLMN are equilateral, and of the same number of sides, they are similar (4), and are to one another as the squares of the diameters of the circles in which they are inscribed; therefore as the square

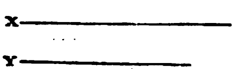


on AD to the square on GL, so is the polygon ABCDEF to the polygon GHKLMN; but as the square on AD to the square on GL, so is the circle ABD to the space Q; therefore the polygon ABCDEF is to the polygon GHKLMN as the circle ABD to the space Q; but the polygon ABCDEF is less than the circle ABD, therefore GHKLMN is less than the space Q (V. 14); wherefore the space Q is greater than any polygon inscribed in the circle GHL.

In the same manner it is demonstrated that Q is less than any polygon described about the circle GHL; wherefore the space Q is equal to the circle GHL (4, Cor. 2). Now, by hypothesis, the circle ABD is to the space Q as the square on AD to the square on GL; therefore the circle ABD is to the circle GHL as the square on AD to the square on GL.

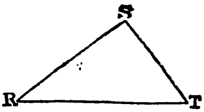
Cor. 1.—Hence the circumferences of circles are to one another as their diameters.

Given the straight line X equal to half the circumference of the circle ABD, and the straight line Y to half the circumference of the circle GHL; and because the rectangles AO · X and GP · Y are equal to the circles ABD and GHL (7); therefore the rectangle AO · X is to the rectangle GP · Y



as the square on AD to the square on GL, or as the square on AO to the square on GP; therefore alternately the rectangle AO · X is to the square on AO as the rectangle GP · Y to the square on GP; but rectangles that have equal altitudes are as their bases, therefore X is to AO as Y to GP; and again, alternately, X is to Y as AO to GP, and taking the doubles of each, the circumference ABD is to the circumference GHL as the *diameter AD to* the diameter GL.

COR. 2.—The circle that is described upon the side of a right-angled triangle opposite to the right angle, is equal to the two circles described on the other two sides; for the circle described upon SR is to the circle described upon RT as the square on SR to the square on RT; and the circle described upon TS is to the circle described upon RT as the square on ST to the square on RT; wherefore the circles described on SR and ST are to the circle described on RT as the squares on SR and ST to the square on RT (V. 24); but the squares on RS and ST are equal to the square on RT; therefore the circles described on RS and ST are equal to the circle described on RT.

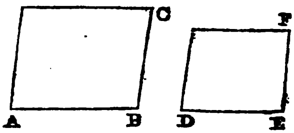


PROPOSITION IX. THEOREM.

Equiangular parallelograms are to one another as the products of the numbers proportional to their sides.

Let AC and DF be two equiangular parallelograms, and let M, N, P, and Q, be four numbers, such that AB : BC :: M : N; AB : DE :: M : P, and AB : EF :: M : Q, and therefore, by equality, BC : EF :: N : Q. The parallelogram AC is to the parallelogram DF as MN to PQ.

Let NP be the product of N into P, and the ratio of MN to PQ will be compounded of the ratios of MN to NP, and of NP to PQ; but the ratio of MN to NP is the same with that of M to P, because MN and NP are equimultiples of M and P; and for the same reason the ratio of NP to PQ is the same with that of N to Q; therefore the ratio of MN to PQ is compounded of the ratios of M to P, and of N to Q. Now, the ratio of M to P is the same with that of the side AB to the side DE; and the ratio of N to Q the same with that of the side BC to the side EF; therefore the ratio of MN to PQ is compounded of the ratios of AB to DE, and of BC to EF; and the ratio of the parallelogram AC to the parallelogram DF is compounded of the same ratios (VI. 23); therefore the parallelogram AC is to the parallelogram DF as MN, the product of the numbers M and N, to PQ the product of the numbers P and Q.



COR. 1.—Hence, if GH be to KL as the number M to number

N, the square described on GH will be to the square described on KL as MM the square of the number M to NN the square of the number N.

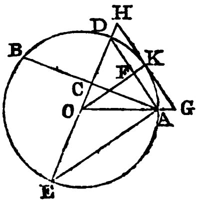
COR. 2.—If the square on GH be to the square on KL as a number to a number, the line GH will be to the line KL as the square root of the first number to the square root of the second number; for, if GH were to KL in any other ratio, the square described on GH would not be to the square described on KL in the ratio supposed.

PROPOSITION X. THEOREM.

The rectangle under the radius of a circle, and the sum of the diameter and supplemental chord of an arc, is equal to the square on the supplemental chord of half that arc.

Given ABE a circle, AB any arc, AD its half; AB and AD their chords, and OF and OC perpendiculars upon them from the centre; *to prove that* then $2EC \cdot EO = AE^2$.

(*Const.*) For, produce EC to D. (*Dem.*) And since the angle EAD is a right angle (III. 31), therefore (I. 28) EA is parallel to OF; therefore (VI. 4) the triangles OFD, EAD are similar, and therefore $DE : DO = EA : OF$. But $DE = 2DO$, therefore $EA = 2OF$, or the perpendicular on AD from the centre is half of the supplemental chord EA. Similarly it is shewn that OC is half the supplemental chord of the arc AB. The triangles CEA and EAD are similar, the angles at A and C being right angles, and that at E common; therefore $CE : EA = EA : ED$, and therefore (VI. 16) $CE \cdot ED = EA^2$, or $2CE \cdot EO = EA^2$, but $2CE = 2EO + 2OC = DE + 2OC$, and 2OC is the supplemental chord of AB; therefore $2CE \cdot EO = AE^2$.



COR. 1.—If P be the perpendicular on the chord of an arc, and Q that on the chord of half that arc, and the radius $= 1$; then $4Q^2 = 1(2 + 2P)$, or $2Q^2 = \frac{1}{2}(2 + 2P) = 1 + P$, or $Q^2 = \frac{1}{2}(1 + P)$.

For $AE = 2OF = 2Q$, and $2CE = 2EO + 2OC = 2 + 2P$, and $EO = 1$.

COR. 2.—Let C denote the chord on which P is the perpendicular, then $\frac{1}{4}C^2 = 1 - P^2$.

For if $AD = C$, $AD^2 = 4AF^2$, and $AF^2 = OA^2 - OF^2$.

COR. 3.—Let R = the side of an inscribed regular polygon, and S that of the corresponding circumscribed one, and P the perpendicular on the former; then $S = \frac{R}{P}$.

For, if AD, GH be the sides of the polygons, then $OF : OK = AD : GH$, or $P : 1 = R : S$, therefore $P \cdot S = R$ (VI. 16), or $S = \frac{R}{P}$.

COR. 4.—The triangles DCA, DAE are similar, and therefore $ED : DA = DA : DC$, or $AD^2 = ED \cdot DC = ED(OD - OC) = 2(1 - OC)$.

Schol.—In these four corollaries and the next proposition, the radius is supposed = 1, and numbers proportional to the other lines are taken instead of the lines. When the square of a number is found, the number itself can of course be found by extracting the square root of the former.

PROPOSITION XI. PROBLEM.

To find the approximate ratio of the diameter of a circle to its circumference as nearly as may be required.

This may be done by calculating first the apothem of an inscribed regular polygon, by taking some polygon, the length of the side of which is accurately known, as that of a square or hexagon. Then calculate (10, Cor. 1) the apothems of the polygons found by doubling successively the number of sides, till at last the apothem of a polygon of a sufficient number of sides be found; its side may then be found (10, Cor. 2); and then the side of the corresponding circumscribed polygon may be found (10, Cor. 3). The perimeters of the two last polygons will be found by multiplying one of their sides by the number of sides, and it will be found that the numbers expressing their values are the same for a certain number of places of figures; and since the value of the circumference of the circle is intermediate between these, it will be accurately expressed to that number of places by the figures common to both these former numbers; and this approximate value may, by the same method, be carried to any degree of accuracy required.

If a hexagon be taken for the first polygon, its side $R = 1$, therefore the square of its apothem is $= 1 - \frac{1}{4}R^2 = 1 - \frac{1}{4} = \frac{3}{4}$; and if its apothem be called A, $A^2 = \frac{3}{4}$. If B be the apothem on the polygon of 12 sides, then (10, Cor. 1) $2B^2 = 1 + A$, and as A is found, B^2 and therefore B will be found. If C be the apothem on the inscribed polygon of 24 sides, then similarly $2C^2 = 1 + B$, and if B be known, C^2 and therefore C can be found. In the same manner, the values of D, E, F, &c., the apothems of inscribed polygons of 48, 96, 192, &c., sides may be found. The following are the values of the apothems of eight successive polygons, beginning with the hexagon:

No. of Sides of Polygon.		Apothem.	Value of Apothem.
6	$A^2 = 1 - \frac{1}{4} = \frac{3}{4}$ and	A =	·86602540
12	$2B^2 = 1 + A$	B =	·96592583
24	$2C^2 = 1 + B$	C =	·99144486
48	$2D^2 = 1 + C$	D =	·99785892
96	$2E^2 = 1 + D$	E =	·99946458
192	$2F^2 = 1 + E$	F =	·99986614
384	$2G^2 = 1 + F$	G =	·99996653
768	$2H^2 = 1 + G$	H =	·99999163
1536	$2K^2 = 1 + H$	K =	·99999791

Now (10, Cor. 4) if AB be the side of the polygon of 768 sides, then $OC = H$, and $AD^2 = 2(1 - H)$, and $AD = {}$·004090612, which is the side of the inscribed polygon of 1536 sides. But (10, Cor. 3) if $AD = R$, and $GH = S$, then $S = \frac{R}{K} = {}$·004090618 = the side of the corresponding circumscribing polygon. And the perimeters of these two polygons, found by multiplying these two sides by 1536, are respectively 6·283180032 and 6·283189248; and consequently the approximate value of the circumference of the circle, carried to 6 places in the decimal part, is 6·283185.

If the first inscribed regular polygon be a square, instead of a hexagon, its apothem is the square root of $\frac{1}{2}$ or of ·5, and the apothems and sides of the successive inscribed and circumscribed polygons of 8, 16, 32, &c., may be calculated in the same manner as above. If the sides of these polygons be found, and then their *areas be calculated* by multiplying their perimeter by half

their apothem, these areas carried to 7 places in the decimal, will be found to be the same as in the subjoined table:

Number of Sides.	Area of Inscribed Polygon.	Area of Circumscribed Polygon.
4	2 · 0000000	4 · 0000000
8	2 · 8284271	3 · 3137085
16	3 · 0614674	3 · 1825979
32	3 · 1214451	3 · 1517249
64	3 · 1365485	3 · 1441184
128	3 · 1403311	3 · 1422236
256	3 · 1412772	3 · 1417504
512	3 · 1415138	3 · 1416321
1024	3 · 1415729	3 · 1416025
2048	3 · 1415877	3 · 1415951
4096	3 · 1415914	3 · 1415933
8192	3 · 1415923	3 · 1415928
16384	3 · 1415925	3 · 1415927
32768	3 · 1415926	3 · 1415926

Since the areas of the last inscribed and circumscribed polygons agree as far as the seventh decimal place, this must be also the area of the circle, which is always of an intermediate value; and since the area of a circle is equal to the rectangle under its circumference and half its radius, therefore the radius being 1, the value of the semicircumference is 3·1415926, which is also the ratio of the circumference of any circle to its diameter, carried to the seventh decimal place.

The value of this ratio may be carried in the same manner to any required degree of approximation; but much more expeditious methods of effecting this are afforded by analytical principles. This approximate value was found by Archimedes to be $3\frac{1}{7}$, or 22 to 7; by Peter Metius 355 to 113; it was carried by Vieta to 11 figures; by Adrianus Romanus to 17; by Ludolph Van Ceulen to 36; by Abraham Sharp to 74; by Machin to 100; and by De Lagny to 127 figures. This ratio, carried to 36 figures, is 3·14159,26535,89793,23846,26433,83279,50288.

Dr Rutherford has since carried it to 208 figures.

GEOMETRICAL MAXIMA AND MINIMA.

DEFINITIONS.

1. Of all magnitudes that fulfil the same conditions, the greatest is called a *maximum*, and the least a *minimum*.

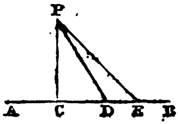
2. Figures that have equal perimeters are said to be *isoperimetrical.*

AXIOM.

Of all isoperimetrical figures, there must be at least one such that its area is not exceeded by that of any other, and if there be only one such, it is the maximum.

PROPOSITION I. THEOREM.

Of all straight lines that can be drawn from a given point to a given straight line, the perpendicular is the least.



Given the point P, and the line AB; draw PC perpendicular to AB, and any other line PD meeting AB; *to prove that* $PC \angle PD$.

(*Dem.*) For since the angles at C are right angles, angle PDC is less than a right angle (I. 17); therefore $PC \angle PD$.

COR.—A line nearer to the perpendicular is less than one more remote.

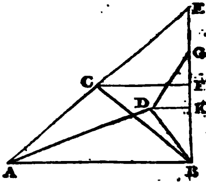
For angle PDE is greater than a right angle (I. 16), and PEC is less (I. 17), therefore $PD \angle PE$.

PROPOSITION II. THEOREM.

Of all triangles having the same base and equal perimeters, that is a maximum whose undetermined sides are equal.

Given two triangles ACB, ADB, having the same base AB, and $AC + CB = AD + DB$; *to prove that* if $AC = CB$, the triangle $ACB > ADB$.

(*Const.*) For draw BE perpendicular to AB, and draw CF, DK perpendicular to BE, make $FE = FB$ and $KG = KB$, join CE and DG. (*Dem.*) Then the triangles CFE, CFB have $BF = FE$ and FC common, and the angles at F right angles, therefore $CE = CB$. For a similar reason, $DG = DB$. Therefore $AC + CE = AC + CB$, and $AD + DG = AD + DB$. Now angle $ECF = BCF = CBA$ (I. 29) $= CAB$ (I. 5), therefore $ECF + ACF = CAB + ACF =$ two right angles (I. 29), therefore AC and CE are in the same straight line. But $AD + DG > AG$, if AG were joined, therefore $AG < AE$, and hence the point G is nearer to the perpendicular AB than E is (1, Cor.), hence $GB < EB$, and therefore $KB < FB$. Therefore the altitude of the triangle ADB being less than that of ACB, $ADB < ACB$ (VI. 1, Cor. 2).

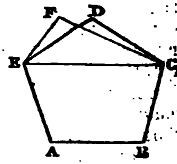


PROPOSITION III. THEOREM.

Of all isoperimetrical polygons of the same number of sides, the equilateral polygon is a maximum.

(*Const.*) Let ABCDE be the maximum polygon, then if two of its sides as ED, DC be not equal, let EF, FC be drawn equal.

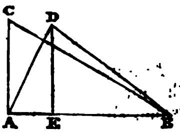
(*Dem.*) Then the triangle $EFC > EDC$ (2); and therefore the given polygon is less than ABCFE, which is contrary to hypothesis. Therefore $ED = DC$; and it may be similarly proved that any two adjacent sides are equal.



PROPOSITION IV. THEOREM.

Of all triangles having only two sides given, that is the greatest in which these sides are perpendicular.

Given two triangles ABC and ABD having the common base AB and $AC = AD$; *to prove that* if AC be perpendicular to AB, the triangle $ABC > ABD$.

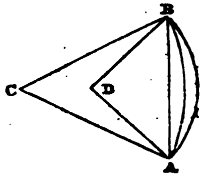


(*Const.*) For draw DE perpendicular to AB. (*Dem.*) Then (1) $DE < AD$, and therefore DE

$\angle$ AC; hence the triangle ACD $>$ ADB (VI. 1, Cor. 2).

PROPOSITION V. THEOREM.

Of all the angles subtended at the centres of different circles by equal chords, that in the least circle is the greatest.

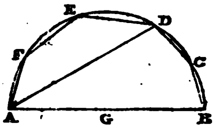


(*Dem.*) Since the same chord AB subtends the angles D and C at the centres of two circles, of which CB and DB are the radii, then the angle D $>$ C (I. 21).

PROPOSITION VI. THEOREM.

There is only one way of forming a polygon, all of whose sides are given, except one, and inscribed in a semicircle, of which the unknown side is a diameter.

Let the *sides* of the polygon ABCDEF be all *given* except AB, and inscribed in a semicircle AEB, of which AB is the diameter. If now a greater circle were taken, the angles at the centre subtended by the chords BC, CD, DE, EF, and FA would be respectively less (Prop. 5) than the angles subtended by them at G, the centre of AEB; that is, they would be less than two right angles; and therefore the extremities A, B of the given sides would not fall at the extremities of a diameter. If a less circle were taken, the sum of the angles would exceed two right angles, and the same consequence would follow. Hence the polygon in question can be inscribed in only one semicircle.



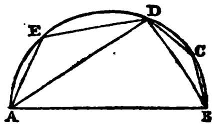
Schol.—The order of the sides AF, FE, &c., may be altered in any manner, the diameter AB and the area of the polygon remaining the same. For the segments cut off by these sides are always the same, and the area of the polygon is equal to that of the semicircle diminished by these segments.

PROPOSITION VII. THEOREM.

Among all polygons whose sides are all given but one, that is, the maximum whose sides can be inscribed in a semicircle, of which the unknown side is the diameter.

Let ABCDE be the greatest polygon, which can be formed

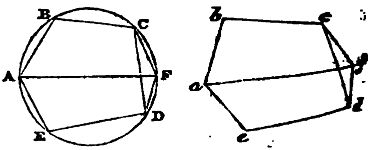
with the given sides AE, ED, DC, and CB, and the last AB assumed of any magnitude. (*Const.*) Join AD and DB. (*Dem.*) Then if the angle ADB were not a right angle, by making it so the triangle ADB would be increased (4), and the parts AED, DCB remaining the same, the whole polygon would be increased. But the given polygon, being a maximum, cannot be augmented; therefore the angle ADB must be a right angle. In the same manner the angles AEB, ACB must be right angles, and the polygon is inscribed in a semicircle.



PROPOSITION VIII. THEOREM.

Of all polygons formed with given sides, that which can be inscribed in a circle is a maximum.

Given two polygons, ABCDE and *abcde*, whose corresponding sides are equal; namely, AB = *ab*, BC = *bc*, &c., and let the former be inscribed in a circle, but the latter incapable of being so inscribed; *to prove that* the former is the greater.



(*Const.*) Draw from A the diameter AF; join CF and FD, and make the triangle *cfd* = CFD in every respect, so that *cf* = CF, *df* = DF, and join *af*.

(*Dem.*) The polygon ABCF $>$ *abcf* (7), unless the latter can be inscribed in a semicircle having *af* for its diameter, in which case the two polygons would be equal. For a similar reason the polygon AEDF $>$ *aedf*, unless in the case of a similar exception, by which the polygons would be equal. Hence the whole polygon ABCFDE $>$ *abcfde*, unless the latter can be inscribed in a circle. But it cannot; therefore the former is the greater. Take from both the equal triangles CFD, *cfd*, and there remains the polygon ABCDE $>$ *abcde*.

Schol.—It may be shewn, as in the sixth proposition, that there is only one circle in which the polygon can be inscribed, and therefore only one maximum polygon; and this polygon will have the same area in whatever order the sides be arranged.

PROPOSITION IX. THEOREM.

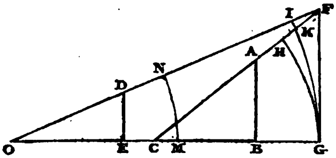
Of all isoperimetrical polygons, having the same number of sides, the regular polygon is the greatest.

(*Dem.*) For (3) the maximum polygon is equilateral, and (8) it can be inscribed in a circle, it is therefore a regular polygon.

PROPOSITION X. THEOREM.

Of two isoperimetrical regular polygons, the one having the greater number of sides is the greater.

Let DE and AB be half-sides of these polygons, OE and CB their apothems, lying in the same straight line, and ACB and DOE the half-angles at their centres. (*Const.*) Since these angles are unequal, the sides OD, CA, if produced, will meet in some point F. Draw FG perpendicular to CG, and from the centre O describe the arcs GI and MN, with the radius OG and the radius $OM = CG$; and describe the arc GH from the centre C with the radius CG.



(*Dem.*) The polygons being regular, their perimeters are the same multiples of their half-sides that four right angles is of their half-angles at their centres, but their perimeters are equal; therefore by direct equality, angle $O : C = DE : AB$. But $O : C = MN : GH$ (VI. 33), therefore $DE : AB = MN : GH$; but $OG : OM$ or $CG = GI : MN$ (Qu.* VIII. Cor. 1), therefore (VI. 23, Cor. 1) $DE \cdot OG : AB \cdot CG = MN \cdot GI : MN \cdot GH = GI : GH$ (VI. 1) if MN be the altitudes of these two rectangles. But the triangles OED and OGF are similar, therefore $OE : OG = DE : FG$ (VI. 4), therefore $OE \cdot FG = DE \cdot OG$ (VI. 16). Also, from the similar triangles ABC and FGC, $CB : AB = CG : FG$; and hence $CB \cdot FG = AB \cdot CG$. Therefore $OE \cdot FG : CB \cdot FG = GI : GH$, or (VI. 1) $OE : CB = GI : GH$. But (Qu. II. Cor. 2) $GK > GH$, therefore GI is still $> GH$, therefore $OE > CB$. But the perimeters of the polygons being equal, their areas will be proportional to their apothems; therefore the polygon, of which OE is the apothem, is the greater. But its half-angle O is less than that of the

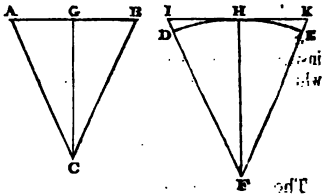
* *Qu. refers to* the preceding book on the Quadrature of the Circle.

other, or it has a greater number of sides; therefore the polygon having the greater number of sides is the greater.

PROPOSITION XI. THEOREM.

The circle is greater than any polygon of the same perimeter.

Let AG be a half-side, and C the centre of a regular polygon of the same perimeter as the circle, of which F is the centre, and DE an arc subtending an angle DFE equal to the angle ACB subtended by the side AB;



then the triangle ACB and the sector DFE are equal parts of the whole polygon and circle, and AB and DE are equal parts of their perimeters, and are therefore equal; therefore if the polygon and circle be respectively denoted by P and C, $P : C =$ triangle $ACB :$ sector $DFE = AG \cdot GC : DH \cdot HF = GC : HF$. But if IHK be a side of the corresponding circumscribed polygon, then by the similar triangles ABC and IFK, $GC : HF = AB$ or $DE : IK$; and therefore $P : C = DE : IK$. But (Qu. II Cor. 2) $DH \angle IH$, and therefore $IK > DE$, hence $C > P$. Now P is a regular polygon, and of all isoperimetrical polygons having the same number of sides, the regular polygon is the greatest; but the circle has been proved to be greater than this polygon; it is therefore greater than any isoperimetrical polygon of equal perimeter.

PROPOSITION XII. THEOREM.

Of all polygons having the same area and the same number of sides, the regular polygon has its perimeter a minimum.

Let A and P be the area and perimeter of a regular polygon B, and N the number of its sides, and A', P', N' those of an irregular polygon B', if $A = A'$ and $N = N'$, then $P \angle P'$.

For let B'' be a polygon similar to B' and A'', P'' and N'' its area, perimeter, and the number of its sides. If $P'' = P$ and $N'' = N$, then $A'' \angle A$ (9); and therefore if the parts of B'' be proportionally increased till $A'' = A$, then its perimeter $= P'$ (*VI. 20*). But $P' > P''$, and therefore $P' > P$.

PROPOSITION XIII. THEOREM.

Of regular polygons having the same area, that which has the greatest number of sides has the least perimeter.

Let B be a regular polygon, A, P, N its area, perimeter, and the number of its sides; A′, P′, N′ those of another regular polygon B′; and A″, P″, N″ those of another B″ similar to B′.
If N $>$ N′, then P is $<$ P′. For since N″ $<$ N, if P″ $=$ P, A″ $<$ A (10); and if the parts of B″ be increased proportionally till A″ $=$ A, then (VI. 20) its perimeter will $=$ P′;
but P′ $>$ P″ and P″ $=$ P, therefore P′ $>$ P.

***Schol.*—In accordance with this and other propositions, bees instinctively construct their cells in the form of regular hexagons, which form requires the least quantity of wax.**

PROPOSITION XIV. THEOREM.

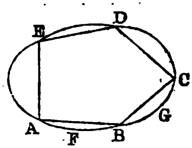
The perimeter of a circle is less than that of any polygon having the same area.

Let A and P be the area and perimeter of a circle B; A′ and P′ those of a polygon B′; and A″, P″ those of a polygon B″ similar to B′. Then if P″ $=$ P, A″ will be $<$ A (11); and if the parts of B″ be proportionally increased till A″ $=$ A,
then its perimeter will $=$ P′; but P′ $>$ P″ and P″ $=$ P, therefore P′ $>$ P.

PROPOSITION XV. THEOREM.

Of all isoperimetrical plane figures, the circle contains the greatest area.

Let ACE be the maximum figure for a given perimeter. If it is not a circle, let an equilateral polygon be described in it.
It is evidently possible for such a polygon to exist in it, so that each of its angular points shall not be in the circumference of the same circle with all the other angular points, and therefore the polygon will be irregular.
Let a regular polygon P, of the same number of sides, and the same perimeter, be described, then each of its sides will be equal to those of the inscribed polygon.



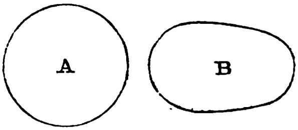
On the sides of P describe segments equal to AFB, BGC, &c., then a curvilinear figure Q will thus be *formed*, having the same perimeter as the given figure

ACE; and since the regular polygon exceeds the other polygon (9), the whole figure Q will exceed the whole ACE. When ACE therefore is not a circle, another figure as Q, having the same perimeter, can always be found greater than it; hence the circle is the maximum figure (Ax.).

Schol.—No part of the figure ACE is supposed to be convex internally. Were any portion of it convex, an equal concave arc being substituted for it, the area would thus be increased, while the perimeter is unaltered.

PROPOSITION XVI. THEOREM.

Of all plane figures of the same area, the circle has the least perimeter.



Let the circle A = the figure B, the perimeter of A is less than that of B. For, if not, let another circle C be assumed whose perimeter is equal to that of B; then (15) its area exceeds that of B or of A; and hence its perimeter exceeds that of A (Qu. VIII.).

EXERCISES.

1. A straight line and two points without it being given, to find a point in it such, that the sum of the lines drawn from it to the two given points shall be a minimum; that is, that they shall be less than the sum of any two lines similarly drawn from any other point in it.

2. If an eccentric point be taken in the diameter of a circle, of all the chords passing through this point, that is the least which is perpendicular to the diameter.

3. Of all triangles that have the same vertical angle, and whose bases pass through a given point, that whose base is bisected by the point is a minimum.

4. The sum of the four lines drawn to the angular points of any quadrilateral from the intersection of the diagonals, is less than that of any other four lines similarly drawn from any other point.

5. To find a point in a given line such, that the difference of the lines drawn to it from two given points may be a maximum.

6. A straight line, and two points on the same side of it, being given, to find a point in it such, that the angle contained by lines drawn from it to the given points shall be a maximum.

7. The sum of two lines drawn from two given points to a point in the circumference of a given circle, is least when these lines are equally inclined to the radius or the tangent drawn to that point.

8. Given three points, to find a fourth such, that the sum of its distances from the given points shall be a minimum.

9. Given the base and altitude of a triangle, to construct it so that the sum of the squares on its sides may be a minimum.

10. Given the perimeter of a parallelogram, to construct it so that its area may be a maximum.

GEOMETRICAL ANALYSIS.

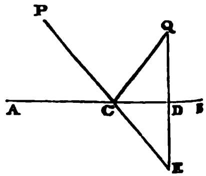
In the method of Geometrical Analysis, the process of demonstration follows an order the reverse of that observed in the ordinary or Synthetic Method. The latter method proceeds from established principles, and, by a chain of reasoning, deduces new principles from these; the former proceeds from the principles that are to be established considered as known, and from these, taken as premises, arrives, by reversing the chain of reasoning, at known principles. The latter method is the didactic method used in communicating instruction; the former is rather employed in the discovery of truth.

Magnitudes are said to be *given* when they are actually given, or may easily be found; they are said to be *given in position* when their position may be determined; and rectilineal figures are said to be *given in species* when figures similar to them may be found. A circle is said to be given *in position* when its centre is given, and *in magnitude* when its radius is given. A *ratio* is said to be *given* when two quantities having that ratio are given.

The two following propositions are given as examples of this method.

PROPOSITION I. PROBLEM.

Given two points P and Q and a straight line AB, *to find* a point C in AB such, that lines PC and CQ, drawn to it from P and Q, may make equal angles PCA and QCD, with AB.



By analysis.—From either of the two given points as Q, draw QD perpendicular to AB, and produce QD to meet PC produced in E; then the angle ECD = ACP (I. 15) = QCD (Hyp.), and the angles CDE and CDQ are equal, being right angles, and CD is common to the two triangles CDE and CDQ; therefore (I. 26) they are equal in every respect, and hence DE = DQ. But the perpendicular DQ is given, and therefore DE and the point E are given; hence the line PE, and consequently the point C, are given.

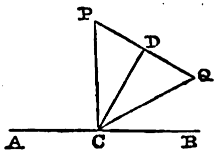
Schol.—By this analysis, the construction is discovered by which the point C is determined. That C is the required point may now be demonstrated by synthesis or composition, thus:

By composition.—(*Const.*) Draw QD perpendicular to AB, and produce DQ till DE = DQ, and join PE, then C will be the point required. Join also CQ.

(*Dem.*) Then because QD = DE, and DC is common to the two triangles CDE and CDQ, and the angles CDE and CDQ are equal, being right angles, therefore (I. 4) these triangles are every way equal; and hence angle ECD = QCD. But PCA = ECD (I. 15), therefore also PCA = QCD; and C is the point required.

PROPOSITION II. PROBLEM.

A straight line AB, and two points P and Q without it being given, to find a point C in it such, that the two lines CP and CQ drawn to them from C shall be equal.



Analysis.—Join PQ, and bisect it in D, and join CD. Because CP = CQ, and PD = DQ, and CD is common to the two triangles CDP, CDQ, they are every way equal (I. 8), therefore the angle PDC = QDC. Therefore CD is perpendicular to PQ. But PQ is given, therefore the point D and the perpendicular DC are given, and consequently the point C is given.

It may be easily proved by composition that C is the point required.

Any of the exercises given in the preceding books will serve as exercises in Geometrical Analysis.

PLANE LOCI.

If the position of a point is determined by certain conditions; also, if every point in some line, and no other point fulfil these conditions, the line is said to be a *locus* of the point.

As a simple illustration of a locus, consider that of a point which is always equally distant from a given point. This is obviously a circle, whose radius is equal to that distance. So the locus of a point, which is always equally distant from a given straight line, is a line parallel to it, and at a distance from it equal to the given distance.

EXERCISES.

1. Find the locus of the vertices of all the triangles that have the same base and one of their sides of a given length.

2. Find the locus of a point that is at equal distances from two given points.

3. Find the locus of a point that is equally distant from two given lines, either parallel, or inclined to one another.

4. To find the locus of the vertices of all the triangles that have the same base and equal altitudes.

5. To find the locus of the vertices of all the triangles that have the same base and one of the angles at the base the same.

6. To find the locus of the angular point opposite to the hypotenuse of all the right-angled triangles that have the same hypotenuse.

7. To find the locus of the vertices of all the triangles that have the same base and equal vertical angles.

8. To find the locus of the vertices of all the triangles that have the same base and equal areas.

9. To find the locus of the vertices of all triangles that have the same base and the sum of the squares of their sides equal to a given square.

10. If straight lines drawn from a given point to a given line be cut in a given ratio, to find the locus of the point of section.

11. To find the locus of the vertices of all the triangles that have the same base and the ratios of their sides equal.

12. If a straight line drawn from a given point, and terminating in the circumference of a given circle, be cut in a given ratio, the locus of the point of section is also the circumference of a given circle.

13. If one of the extremities of straight lines drawn through a given point be terminated in a given straight line, and their other extremities be determined, so that the rectangle under the segments of each line is equal to a given rectangle, the locus of these extremities will be the circumference of a circle; and if one of the extremities be terminated in the circumference of a circle, the locus of the other extremities is a straight line.

14. If the vertex of a triangle be fixed, and its altitude given, find the loci of the extremities of its base, so that the area may be constant.

PORISMS.

A *porism* is a proposition of an indeterminate nature, such that an indefinite number of quantities must fulfil the same conditions. As a simple example of a porism, let it be required to find a point such, that all straight lines drawn from it to the circumference of a given circle shall be equal. This point is evidently the centre of the circle. Problems of loci and porisms are in many cases mutually convertible. The preceding problem becomes a problem of the latter kind when the centre is given, and it is required to find the locus of all the points that are at a given distance from it.

EXERCISES.

1. Two points being given, to find a third, through which any straight line being drawn, the perpendiculars upon it from the two given points shall be equal.

2. Three points being given, to find a fourth, through which any straight line being drawn, the sum of the perpendiculars upon it from two of the given points on one side of it, shall be equal to the perpendicular on it from the third.

There are similar porisms when four, five, or any number of points are given, by which another point is to be found such, that any straight line being drawn through it, the sum of the perpendiculars from the points on one side of it shall be equal to the sum of those from the points on the other side of it. The point so found is called the point of mean distance; or if the points be considered to be equal, physical, or material points, it is called the centre of gravity.

3. A straight line and a circle being given, to find a point such, that the rectangle under the segments of any straight line drawn through it, and limited by these, shall be equal to the rectangle contained by the external segment of a diameter perpendicular to the line, and produced to meet it, and the diameter itself.

This problem is the 13th exercise on loci converted into a porism. Many other problems of loci may be similarly converted; and conversely.

PLANE TRIGONOMETRY.

Plane Trigonometry treats of those relations that subsist between the sides and angles of triangles, by which their numerical values may be computed.

These relations are established by means of certain lines connected with the angles, called trigonometrical lines. When a sufficient number of the parts of a triangle are known, it may be constructed, and then the unknown parts can be measured. This method, however, which is called construction, or the graphic method, would only give a moderate approximation; but, by trigonometrical computation, the values may be found to any required degree of accuracy. The sides are measured by some line of a determinate length, chosen as the unit of measure—as a foot, a yard, a mile, &c.; and the unit of measure of angles is the 90th part of a right angle, or the 360th part of four right angles. As the angles at the centre of a circle are proportional to the arcs subtending them, these arcs may be taken as their measures. The circumference of a circle is accordingly supposed to be divided into 360 equal parts, called degrees; each of these into 60 equal parts, called minutes; each of these into 60 equal parts, called seconds; and so on. If, therefore, a circle be described from the angular point as a centre, the number of degrees, minutes, &c., contained in the arc intercepted by the lines containing the angle, is the measure of that angle. An angle is also sometimes measured by the length of the intercepted arc of a circle, divided by the radius, whose centre is the angular point in this case, $3{\cdot}1415926 = 180°$.

DEFINITIONS OF TRIGONOMETRICAL LINES.

1. The *complement* of an arc is its difference from a quadrant; and that of an angle its difference from a right angle.

2. The *supplement* of an arc is its defect from a semicircle; and that of an angle is its defect from two right angles.

3. The *sine* of an arc is a line drawn from one of its extremities, perpendicular to the radius passing through its other extremity.

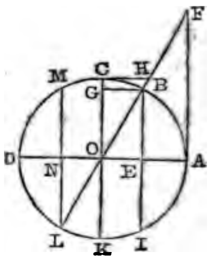
4. The *tangent* of an arc is a line touching it at one extremity, and limited by the radius produced through its other extremity.

5. The *secant* of an arc is the length of the line, which is intercepted between the extremity of the tangent and the centre.

6. The *versed sine* of an arc is that portion of the radius intercepted between the sine and the extremity of the arc.

7. The *supplemental versed sine*, or *suversed sine*, is the difference between the versed sine and the diameter.

8. The sine, tangent, secant, &c., of the complement of an arc, are concisely termed the *cosine*, *cotangent*, *cosecant*, &c., of that arc. These terms, for conciseness, are usually contracted into sin., tan., sec., vers., suvers., cos., cot., cosec., covers., and cosuvers.



Let AB be an arc of a circle, AC a quadrant, O the centre; BE and BG perpendiculars on the radii OA and OC; AF and CH tangents at A and C; then BC is the complement of AB, and COB that of angle AOB; DCB the supplement of AB, and angle DOB that of AOB; BE is the sine of AB; AF its tangent; OF its secant; AE its versed sine; DE its suversed sine; also BG, CH, and OH are the sine, tangent, and secant of BC, or the cosine, cotangent, and cosecant of AB.

COROLLARIES FROM THE DEFINITIONS.

1. The sine of a quadrant, or of a right angle, is equal to the radius.

For CO = sin. AC.

2. The tangent of half a right angle is equal to the radius.

For if angle AOF were half a right angle, so would AFO (I. 32), and therefore AF would be equal to AO.

3. The sine, tangent, &c., of an arc are equal in length to those of its supplement.

For BE is the sine of DCB; AF is the tangent of AKL, and therefore of the equal arc DCB; and OF is its secant.

4. The radius is a mean proportional between the tangent and cotangent of an arc.

For FAO, OCH are similar triangles, and therefore AF : AO = OC : CH.

If the arc AB be called A, and the radius OA be called R, then tan. A : R = R : cot. A, or R^2 = tan. A · cot. A.

5. The radius is a mean proportional between the sine and cosecant of an arc.

From the similar triangles BEO and OCH, $BE : BO = OC : OH$, or $\sin. A : R = R : \text{cosec. } A$, or $R^2 = \sin. A \cdot \text{cosec. } A$.

6. The radius is a mean proportional between the cosine and secant of an arc.

By the similar triangles BGO and OAF, $GB : BO = OA : OF$, or $\cos. A : R = R : \sec. A$, or $R^2 = \cos. A \cdot \sec. A$.

7. The tangent is to the radius as the sine to the cosine.

For in the similar triangles OAF and OEB, $FA : AO = BE : EO$, or $\tan. A : R = \sin. A : \cos. A$.

8. The tangent is to the secant as the radius to the cosecant.

For in the similar triangles FAO and OCH, $FA : FO = CO : OH$, or $\tan. A : \sec. A = R : \text{cosec. } A$.

9. The sine of an arc is to the cosine as the secant to the cosecant.

For (Cor. 5 and 6) $\sin. A \cdot \text{cosec. } A = \cos. A \cdot \sec. A$, therefore (VI. 16) $\sin. A : \cos. A = \sec. A : \text{cosec. } A$.

10. The square on the radius is equal to the squares on the sine and cosine of an arc.

For $OE = GB = \cos. A$, and $OB^2 = BE^2 + EO^2$, therefore $R^2 = \sin. {}^2A + \cos. {}^2A$.

11. The square on the sine of an arc is equal to the difference of the squares on the radius and the cosine.

For (Cor. 10) $\sin. {}^2A + \cos. {}^2A = R^2$, therefore $\sin. {}^2A = R^2 - \cos. {}^2A$.

12. The square on the sine of an arc is equal to the rectangle under the versed sine and suversed sine.

For (III. 35) $BE^2 = AE \cdot ED$, or $\sin. {}^2A = \text{vers. } A \cdot \text{suvers. } A$; since AE is the vers. A and ED is the suvers. A.

13. The square on the secant of an arc is equal to the sum of the squares on the radius and tangent.

For $OF^2 = OA^2 + AF^2$, or $\sec. {}^2A = R^2 + \tan. {}^2A$. Similarly, $\text{cosec. } {}^2A = R^2 + \cot. {}^2A$.

14. The chord of twice an arc is equal to twice the sine of that arc.

For $BI = 2BE$, or chord $2A = 2 \sin. A$.

15. The square on the chord of an arc is equal to twice the rectangle under the radius and the versed sine of the arc.

For (Cor. 12) $BE^2 = AE \cdot ED$, therefore $BE^2 + EA^2 = AE \cdot ED + AE^2$, or $AB^2 = AD \cdot AE$, or chord ${}^2A = 2R. \text{vers. } A$.

16. The sine, tangent, and secant of an arc, are the same as those of that arc increased by any number of whole circumferences.

For if to AB any number of whole circumferences be added, the compound arc will terminate at B.

In some of the applications of trigonometry, it is necessary to attend to the signs of the trigonometrical lines. Quantities whose signs are +, are called *positive;* and those whose signs are −, are called *negative.* If a line be measured from a given point or a given line as its origin, it is reckoned *positive* when it lies on one side of its origin, and *negative* when on the opposite side. Thus, if the sines in the first quadrant be measured from the diameter AD, and be reckoned positive, those in the second quadrant, as MN, will also be positive; and those in the third and fourth, as LN, IE, will be negative. So if the cosines, as BG, in the first quadrant, measured from CK be reckoned positive, those in the second and third, being measured from the opposite side of CK, will be negative, and those in the fourth will be positive. In a similar manner, the signs of the tangent, secant, &c., are determined; but this subject belongs properly to Analytical Trigonometry.

If a number as m is to be divided by a number n, the quotient is expressed thus, $\frac{m}{n}$.

If R = 1, and the whole circumference = C, these corollaries become respectively:

1. Sin. $\frac{1}{4}$C = 1.

2. Tan. $\frac{1}{8}$C = 1.

3. Sin. A = sin. ($\frac{1}{2}$C − A).

4. Tan. A. cot. A = 1, therefore tan. A $= \frac{1}{\text{cot. A}}$, and cot. A $= \frac{1}{\text{tan. A}}$; or the tangent and cotangent are each other's reciprocals.

5. Sin. A. cosec. A = 1, therefore sin. A $= \frac{1}{\text{cosec. A}}$, and cosec. A $= \frac{1}{\text{sin. A}}$; or the sine and cosecant are reciprocals.

6. Cos. A. sec. A = 1, therefore cos. A $= \frac{1}{\text{sec. A}}$, and sec. A $= \frac{1}{\text{cos. A}}$; or the cosine and secant are reciprocals.

7. Tan. A. cos. A = sin. A, or tan. A $= \frac{\text{sin. A}}{\text{cos. A}}$.

8. Tan. A. cosec. A = sec. A, or tan. A = $\frac{\text{sec. A}}{\text{cosec. A}}$.

9. $\frac{\text{Sin. A}}{\text{Cos. A}} = \frac{\text{sec. A}}{\text{cosec. A}}$.

10. Sin.^{2}A + cos.^{2}A = 1.

11. Sin.^{2}A = 1 − cos.^{2}A = (1 + cos. A)(1 − cos. A) (II. 5, Cor.); so cos.^{2}A = 1 − sin.^{2}A = (1 + sin. A) (1 − sin. A).

12. Sin.^{2}A = vers. A. suvers. A.

13. Sec.^{2}A = 1 + tan.^{2}A, and cosec.^{2}A = 1 + cot.^{2}A.

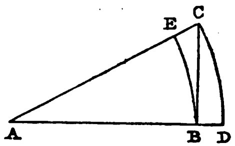
14. Chord 2A = 2 sin. A.

15. Chord ^{2}A = 2 vers. A.

16. Sin. A = sin. (mC + A), where m is any integer.

PROPOSITION I. THEOREM.

If the hypotenuse of a right-angled triangle be made radius, the sides become the sines of the opposite angles, or the cosines of the adjacent angle.



For let ABC be a right-angled triangle, if its hypotenuse AC be made radius, then BC is the sine of A; but the angle at C is the complement of A, therefore BC is the cosine of C. By making C the centre, it may be similarly proved that AB is the sine of C, and therefore the cosine of A.

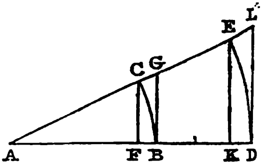
PROPOSITION II. THEOREM.

If one of the sides about the right angle of a right-angled triangle be made radius, the other side becomes the tangent of the opposite angle, and the hypotenuse the secant of the same; or the other side becomes the cotangent of the adjacent angle, and the hypotenuse the cosecant of the same.

When AB (see fig. to Prop. 1) is radius, BC is tangent of A, and AC is secant of A, or BC is cotangent of C, and AC is cosecant of C.

PROPOSITION III. THEOREM.

The sine, tangent, and other trigonometrical lines of an angle for one radius, are proportional to those for another radius; and the ratios of the corresponding lines are the same as that of the respective radii.



Let A be any angle, and BC and DE be arcs described with the radii AB or AC, and AD or AE; then CF and EK are its sines, and BG and DL its tangents for these radii respectively, CF : EK = AC : AE, or the sines of angle A for the radii AC and AE, or AB and AD are proportional to these radii. The same may be proved for the tangents GB and LD, and the secants AG and AL. Hence, by equal ratios, CF : EK = GB : LD = AG : AL.

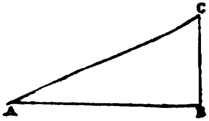
COR.—Hence, in a right-angled triangle AFC, since AC : CF = R : sin. A, R · CF = AC · sin. A, and if R = 1, CF = AC · sin. A.

It is proved in the same way that AF = AC · cos. A; and when AF is radius, FC = AF · tan. A, and AC = AF · sec. A.

Schol.—When two sides of a right-angled triangle are given, the third may be found by I. 47 and Cor.

PROPOSITION IV. THEOREM.

If any side of a right-angled triangle be made radius, the other two sides are proportional to the trigonometrical lines, which they represent for any other radius.

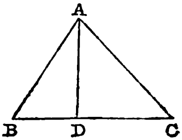


Let ABC be a right-angled triangle; when AC is radius, AB and BC represent the cosine and sine of A; and (Prop. III.) these are proportional to the cosine and sine for any other radius. Therefore if sin. A, cos. A, be the sine and cosine of A for any other radius, CB : BA = sin. A : cos. A; or sin. A : cos. A = CB : BA.

PROPOSITION V. THEOREM.

The sides of a plane triangle are to one another as the sines of the opposite angles.

From A, any angle in the triangle ABC, let AD be drawn perpendicular to BC; and because the triangle ABD is right-angled at D, AB : AD = R : sin. B, or AD · R = AB sin. B; and for the same reason, AC : AD = R : sin. C, or AD · R = AC sin. C; therefore AB · sin. B = AC · sin. C, hence (VI. 16), AB : AC = sin. C : sin. B. In the same manner it may be demonstrated that AB : BC = sin. C : sin. A.

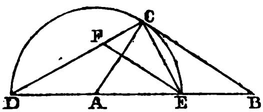


PROPOSITION VI. THEOREM.

The sum of two sides of a triangle is to their difference as the tangent of half the sum of the angles at the base to the tangent of half their difference.

Let ABC be any triangle, then if B and C denote the angles at B and C respectively, AB + AC : AB − AC = tan. $\frac{1}{2}$(C + B) : tan. $\frac{1}{2}$(C − B).

(*Const.*) From A as a centre, with the radius AC, describe the semicircle DCE; produce BA to D; join DC and CE, and draw EF parallel to BC. (*Dem.*) Then twice angle AEC = AEC + ACE = DAC (I. 32) = ABC + ACB = B + C, therefore AEC = $\frac{1}{2}$(C + B); and FEC = ECB, and AEC − ECB = B the less angle, for AC ∠ AB; therefore (II. D. Cor. 2) ECB or CEF = $\frac{1}{2}$(C − B). Also, DB = DA + AB = AB + AC, and BE = AB − AE = AB − AC. And since angle DCE is a right angle (III. 31), if EC be made radius, DC and FC are tangents of the angles AEC and FEC (Def. 4). But (VI. 2) DB : BE = DC : CF; or AB + AC : AB − AC = tan. $\frac{1}{2}$(C + B) : tan. $\frac{1}{2}$(C − B).



PROPOSITION VII. THEOREM.

If a perpendicular be drawn from the vertex upon the base of a triangle, the sum of the segments of the base is to the sum of the two sides as the difference between these sides to the difference between the segments of the base.

For (II. C. Cor. 1) the rectangle under the sum and difference of the sides is equal to that under the sum and difference of the segments of the base, therefore (VI. 16) the above proportion subsists.

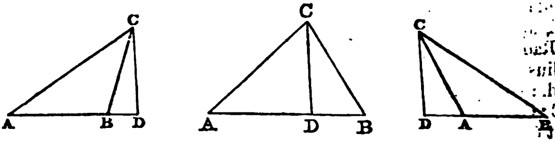
Schol.—The preceding propositions are sufficient for the solution of all the cases of trigonometry; but, in particular cases, some of the following propositions may be employed with advantage.

PROPOSITION VIII. THEOREM.

In any triangle, twice the rectangle contained by any two of its sides, is to the difference between the sum of their squares, and the square of the other side, as radius to the cosine of the angle contained by these two sides.

Given ABC any triangle; *to prove that* $2AB \cdot BC$ is to the difference between $AB^2 + BC^2$ and AC^2 as radius to cos. B.

(*Const.*) For from C draw CD perpendicular to AB or AB produced. Then $BC : BD = R : \cos. B$. But (VI. 1)



$2AB \cdot BC : 2AB \cdot BD = BC : BD$; therefore $2AB \cdot BC : 2AB \cdot BD = R : \cos. B$. But (II. 12, 13, otherwise), $2AB \cdot BD$ is the difference between $AB^2 + BC^2$ and AC^2; and hence the proposition is proved.

COR.—If the sides opposite to the angles A, B, and C be called a, b, and c respectively, and radius = 1, then $2ac : a^2 + c^2 - b^2 = 1 : \cos. B$, or $\cos. B = \dfrac{a^2 + c^2 - b^2}{2ac}$. If the numerical value of the cosine calculated from this expression be positive, the angle B is acute, but if negative, the angle B is obtuse; but the cosine of an obtuse angle is negative, hence, in either case, the above expression gives the true cosine of the angle B. From the above expression, it is evident that the cosine of an angle of a triangle is equal to the sum of the squares of the two sides that contain the angle, diminished by the square of the side opposite the angle, divided by twice the product of the sides which contain the angle,

Hence also $$\cos. A = \frac{b^2 + c^2 - a^2}{2bc},$$

and $$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}.$$

By multiplying both sides of these three values for the three cosines of the angles of a triangle by the denominators of the second side, and, transposing, they become

$$a^2 = b^2 + c^2 - 2bc \cos. A,$$

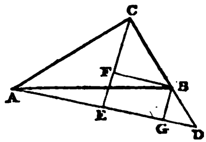
$$b^2 = a^2 + c^2 - 2ac \cos. B,$$

and $$c^2 = a^2 + b^2 - 2ab \cos. C.$$

PROPOSITION IX. THEOREM.

In any triangle, the rectangle under the two sides is to that under the excesses of the semiperimeter above these sides, as the square of the radius to the square of the sine of half the vertical angle.

Let ABC be any triangle, produce CB till CD = CA; join AD, and draw CE bisecting the angle C, then (I. 4) CE is perpendicular to AD. Draw BF parallel to DE, and BG to CE. Then AC : AE = Rad. : sin. $\frac{1}{2}$C (Prop. IV.) if AC be radius; also BC : BF or EG = Rad. : sin. $\frac{1}{2}$C; hence (VI. 23, Cor.) AC $\times$ CB : AE $\cdot$ EG = R^2 : sin.^{2} $\frac{1}{2}$C. But (II. c. Cor. 1) AE $\cdot$ EG = $\frac{1}{2}$(AB + BD)$\frac{1}{2}$(AB − BD) in the triangle ABD.



But AB + BD = AB + CD − CB = AB + AC − CB = AB + AC + CB − 2CB = 2(S − CB), if S = semiperimeter. And AB − BD = AB − (CD − CB) = AB − (AC − CB) = AB − AC + CB = AB + AC + CB − 2AC = 2(S − AC); therefore AC $\cdot$ CB : (S − AC)(S − BC) = R^2 : sin.^{2} $\frac{1}{2}$C.

COR.—If the numerical values of the sides opposite to the angles A, B, and C be called a, b, and c respectively, and if s = the semiperimeter and R = 1, then

$$ab : (s - a)(s - b) = 1^2 : \sin.^2 \tfrac{1}{2}C.$$

or $$ab \cdot \sin.^2 \tfrac{1}{2}C = (s - a)(s - b),$$

or $$\sin.^2 \tfrac{1}{2}C = \frac{(s - a)(s - b)}{ab}.$$

Similarly, $$\sin.^2 \tfrac{1}{2}A = \frac{(s - b)(s - c)}{bc},$$

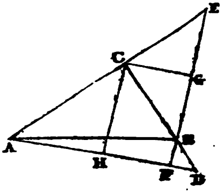
and $$\sin.^2 \tfrac{1}{2}B = \frac{(s - a)(s - c)}{ac}.$$

PROPOSITION X. THEOREM.

In any triangle, the rectangle under its two sides is to that under the semiperimeter, and its excess above the base, as the square of the radius to the square of the cosine of half the vertical angle.

Let ABC be any triangle, and draw the figure ACDH as in the last; produce BF to meet AC produced in E, and draw CG parallel to AD.

Then angle ACH = E, and DCH = CBE, therefore E = CBE, and CE = CB. Also since CG is parallel to AD, the angles at G are right angles. And HG is a rectangle, therefore CH = GF. Now, AC : CH or GF = R : cos. $\frac{1}{2}$C, and CE or CB : EG = R : cos. E or cos. $\frac{1}{2}$C; therefore (VI. 23, Cor.) AC · CB : EG · GF = R^2 : cos.^{2} $\frac{1}{2}$C. But EG · GF = $\frac{1}{2}$EB · $\frac{1}{2}$(EF + FB) = (II. c. Cor. 1) $\frac{1}{2}$(EA + AB) × $\frac{1}{2}$(EA − AB) = $\frac{1}{2}$(BC + CA + AB) × $\frac{1}{2}$(BC + CA + AB − 2AB) = S(S − AB); since EF and FB are the segments of the base of the triangle EAB made by the perpendicular AF from the vertex; therefore AC · CB : S(S − AB) = R^2 : cos.^{2} $\frac{1}{2}$C.



COR.—Take a, b, c, &c., as in Cor. to last proposition, and $ab : s(s-c) = R^2 : \cos.^2 \frac{1}{2}C$; or if R = 1, $ab \cdot \cos.^2 \frac{1}{2}C = s(s-c)$, or

$$\cos.^2 \tfrac{1}{2}C = \frac{s(s-c)}{ab}.$$

Similarly, $\cos.^2 \frac{1}{2}A = \dfrac{s(s-a)}{bc}$, and $\cos.^2 \frac{1}{2}B = \dfrac{s(s-b)}{ac}$.

PROPOSITION XI. THEOREM.

In any triangle, the rectangle under the semiperimeter, and its excess above the base, is to the rectangle under its excesses above the two sides, as the square of the radius to the square of the tangent of half the vertical angle.

For (Prop. X.) AC · CB : S(S − AB) = R^2 : cos.^{2} $\frac{1}{2}$C, and (Prop. IX.) AC · CB : (S − AC)(S − BC) = R^2. sin.^{2} $\frac{1}{2}$C, therefore, by direct equality, S(S − AB) : (S − AC)(S − BC) = cos.^{2} $\frac{1}{2}$C : sin.^{2} $\frac{1}{2}$C. But (Cor. 7 to Definitions, and VI. 22, Cor.)

$\cos.^2 \frac{1}{2}C : \sin.^2 \frac{1}{2}C = R^2 : \tan.^2 \frac{1}{2}C$; therefore $S(S - AB) : (S - AC)(S - BC) = R^2 : \tan.^2 \frac{1}{2}C$.

COR.—Therefore $s(s - c) : (s - a)(s - b) = R^2 : \tan.^2 \frac{1}{2}C$; and when $R = 1$, $s(s - c) \tan.^2 \frac{1}{2}C = (s - a)(s - b)$, or

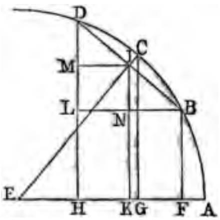
$$\tan.^2 \tfrac{1}{2}C = \frac{(s-a)(s-b)}{s(s-c)}.$$

Similarly, $$\tan.^2 \tfrac{1}{2}A = \frac{(s-b)(s-c)}{s(s-a)},$$

and $$\tan.^2 \tfrac{1}{2}B = \frac{(s-a)(s-c)}{s(s-b)}.$$

PROPOSITION XII. THEOREM.

Let AB, AC, and AD be three such arcs, that BC the difference of the first and second is equal to CD the difference of the second and third; the radius is to the sine of AC, the middle arc, as the cosine of the common difference BC to half the sum of the sines of AB and AD, the extreme arcs.



(*Const.*) Draw CE to the centre, and BF, CG, and DH perpendicular to AE, then BF, CG, and DH are the sines of the arcs AB, AC, and AD. Join BD, and let it meet CE in I; draw IK perpendicular to EA, also draw IM and BL perpendicular to DH.

(*Dem.*) Then, because the arc BD is bisected in C, EC is at right angles to BD, and bisects it in I; also BI is the sine, and EI the cosine of the arc BC or CD. And, since BD is bisected in I, and IM is parallel to BL, LD is bisected in M (2. VI.) and LB is bisected in N. Now BF is equal to LH, therefore $BF + DH = DH + HL = DL + 2LH = 2LM + 2LH = 2HM = 2IK$; and therefore IK is half the sum of BF and DH. But because the triangles ECG and EIK are equiangular, having the angles at G and K right angles and the angle at E common, $EC : CG :: EI : IK$; but it has been shewn that CG is the sine of the arc AC, EI the cosine of the arc BC; therefore $R : \sin.\ AC :: \cos.\ BC : \frac{1}{2}(\sin.\ AB + \sin.\ AD)$.

COR. 1.—Radius is to the cosine of AC, the middle arc, as the cosine of the common difference is to half the sum of the cosines of the extreme arcs.

(*Dem.*) For EH is the cosine of AD, and EF is the cosine of AB, but EH + EF = 2EH + HF = 2EH + 2HK = 2EK, therefore EK = half the sum of the cosines of the extreme arcs.
Now the triangles ECG and EIK are equiangular, hence EC : EG :: EI : EK, that is, R : cos. AC :: cos. BC : $\frac{1}{2}$(cos. AD + cos. AB).

Cor. 2.—Radius is to the sine of AC, the middle arc, as the sine of CD, the common difference, is to half the difference of cosines of the extreme arcs.

(*Dem.*) The difference of the cosines of the extreme arcs is evidently HF = LB = 2LN = 2MI; therefore, MI = half the difference of the cosines of the extreme arcs. Now the triangles BCG and DMI are equiangular, for the angles at G and M are right angles, and the angles DIM and ECG are equal, for they are the complements of the equal alternate angles MIE and IEG, therefore EC : CG :: DI : IM, or R : sin. AC :: sin. CD : $\frac{1}{2}$(cos. AB − cos. AD).

Cor. 3.—Radius is to the cosine of AC, the middle arc, as the sine of DC, the common difference, is to half the difference of the sines of the extreme arcs.

(*Dem.*) The difference of the sines of AD and AB is the difference of DH and BF, which is DH − LH = DL = 2DM; therefore DM is equal to half the difference of the sines of AD and AB. And since it was proved in last corollary that the triangles CEG and IDM are equiangular, EC : EG :: ID : DM, or R : cos. AC :: sin. CD : $\frac{1}{2}$(sin. AD − sin. AB).

Cor. 4.—If AC be considered as one arc, and CB or CD as another, then AD is their sum and AB is their difference; and if AC be called A, and CB or CD be called B, then the arc AD = (A + B), and AB = (A − B); also, if radius = 1, then the proposition becomes

$$1 : \sin. A :: \cos. B : \tfrac{1}{2}\{\sin.(A+B) + \sin.(A-B)\};$$

$$\therefore \tfrac{1}{2}\{\sin.(A+B) + \sin.(A-B)\} = \sin. A \cos. B;$$

similarly, from Cor. 3,

$$\tfrac{1}{2}\{\sin.(A+B) - \sin.(A-B)\} = \cos. A \sin. B;$$

adding and subtracting these, gives

$$\sin.(A+B) = \sin. A \cos. B + \cos. A \sin. B \quad \ldots \quad (A),$$

$$\text{and } \sin.(A-B) = \sin. A \cos. B - \cos. A \sin. B \quad \ldots \quad (B).$$

In (A), let B = A, then $\sin. 2A = 2 \sin. A \cos. A \quad \ldots \quad (C).$

Again, by the same supposition, Cor. 1 becomes

$$1 : \cos. A :: \cos. B : \tfrac{1}{2}\{\cos.(A - B) + \cos.(A + B)\};$$

$$\therefore \tfrac{1}{2}\{\cos.(A - B) + \cos.(A + B)\} = \cos. A \cos. B;$$

and similarly, Cor. 2 becomes

$$\tfrac{1}{2}\{\cos.(A - B) - \cos.(A + B)\} = \sin. A \sin. B;$$

adding and subtracting these, gives

$$\cos.(A - B) = \cos. A \cos. B + \sin. A \sin. B \quad \ldots \quad (D),$$

and $\cos.(A + B) = \cos. A \cos. B - \sin. A \sin. B \quad \ldots \quad (E).$

In (E), let $B = A$, then $\cos. 2A = \cos.^2 A - \sin.^2 A \quad \ldots \quad (F).$

Cor. 5.—Using the same notation as in Cor. 4, the proposition becomes

$$1 : \sin. A :: \cos. B : \tfrac{1}{2}\{\sin.(A + B) + \sin.(A - B)\};$$

therefore $\tfrac{1}{2}\{\sin.(A + B) + \sin.(A - B)\} = \sin. A \cos. B;$

multiplying by 2,

$$\sin.(A + B) + \sin.(A - B) = 2 \sin. A \cos. B;$$

subtracting $\sin.(A - B)$ from both,

$$\sin.(A + B) = 2 \sin. A \cos. B - \sin.(A - B);$$

let $B = 1'$, then

$$\sin.(A + 1') = 2 \sin. A \cos. 1' - \sin.(A - 1') \quad \ldots \quad (G).$$

If now in this formula A be put equal to $1'$, $2'$, $3'$, $4'$, &c., it becomes

$$\sin. 2' = 2 \sin. 1' \cos. 1' - 0 \text{ since } \sin. 0 = 0,$$

$$\sin. 3' = 2 \sin. 2' \cos. 1' - \sin. 1',$$

$$\sin. 4' = 2 \sin. 3' \cos. 1' - \sin. 2',$$

$$\sin. 5' = 2 \sin. 4' \cos. 1' - \sin. 3',$$

$$\sin. 6' = 2 \sin. 5' \cos. 1' - \sin. 4'.$$

In this manner a table of natural sines may be constructed extending through the whole quadrant, if the sine and cosine of one minute can be found. Now, in the Quadrature of the Circle, Proposition x. Cor. 1, it is shewn that $Q^2 = \tfrac{1}{2}(1 + P)$, where if P be the cosine of any arc, Q is the cosine of half the arc, therefore $\cos. \dfrac{A}{2} = \sqrt{\dfrac{1 + \cos. A}{2}}$; now it can easily be shewn that $\cos. 60° = \tfrac{1}{2}$, for Cor. 4 (C), $\sin. 2A = 2 \sin. A \cos. A$; let $A = 30°$,

therefore sin. $60° = 2$ sin. 30 cos. 30, but cos. $30° =$ sin. $60°$, and sin. $30° =$ cos. $60°$, hence sin. $60° = 2$ cos. $60°$ sin. $60°$, and dividing by 2 sin. $60°$, cos. $60° = \frac{1}{2}$; therefore cos. $30°$ can be found, and hence cos. $15°$ can be found. Proceeding in this manner, the cosine of an arc less than one minute may be found; and since (Trig. 10) $\sin.^2 A + \cos.^2 A = 1$, $\sin. A = \sqrt{1 - \cos.^2 A}$, from which the sine of the same arc may be found. Then since the sines of small arcs are very nearly proportional to their sines, the sine of one minute will be found to be $= \cdot 0002908882$, when radius $= 1$. The sine of $1'$ being found, the cos. $1' = \sqrt{1 - \sin.^2 1'}$. In this way the cosine of $1'$ is found to be $= \cdot 9999999577 = 1 - \cdot 0000000423$. Now if this latter form of the cosine of $1'$ be substituted for it in formula (G) above, it becomes

$$\sin. (A + 1') = 2 \sin. A - \cdot 0000000846 \sin. A - \sin. (A - 1');$$

hence $\sin. 2' = 2 \sin. 1' - \cdot 0000000846 \sin. 1' - 0$,

$$\sin. 3' = 2 \sin. 2' - \cdot 0000000846 \sin. 2' - \sin. 1',$$

$$\sin. 4' = 2 \sin. 3' - \cdot 0000000846 \sin. 3' - \sin. 2',$$

$$\sin. 5' = 2 \sin. 4' - \cdot 0000000846 \sin. 4' - \sin. 3', \text{ \&c.}$$

COR. 6.—Using the same notation as in Cor. 4, it follows, from Cor. 3, that $\sin. (A + B) - \sin. (A - B) = 2 \cos. A \sin. B$; in this formula, let $A = 60°$, and remembering that cos. $60° = \frac{1}{2}$, and therefore 2 cos. $60° = 1$, it becomes

$$\sin. (60° + B) - \sin. (60° - B) = \sin. B,$$

or $\sin. (60° + B) = \sin. (60° - B) + \sin. B$,

which shews that when the sines have been calculated up to $60°$, all those after may be found by taking the sine of an arc as much less than $60°$, and adding to it the sine of the arc by which it exceeds $60°$.

As shewn above, a table of the *sines*, and consequently of the *cosines*, of arcs of any number of degrees and minutes, from 0 to $90°$ may be constructed. And since $\tan. A = \frac{\sin. A}{\cos. A}$, the table of *tangents* is computed by dividing the sine of any arc by the cosine of the same arc. Again, since $\sec. A = \frac{1}{\cos. A}$, the *secants* will be found by dividing 1 by the *cosine* of the arc whose secant is sought. Lastly, the *versed sines* are found by subtracting the *cosines* of the arcs from 1.

(1.) Again, resuming the results obtained in (Prop. XII.), and its first, second, and third corollaries, as collected in (Cor. 4); namely,

$$\tfrac{1}{2}\{\sin.(A+B)+\sin.(A-B)\}=\sin. A\cos. B \quad \ldots \quad (a),$$

$$\tfrac{1}{2}\{\sin.(A+B)-\sin.(A-B)\}=\cos. A\sin. B \quad \ldots \quad (b),$$

$$\tfrac{1}{2}\{\cos.(A-B)+\cos.(A+B)\}=\cos. A\cos. B \quad \ldots \quad (c),$$

$$\tfrac{1}{2}\{\cos.(A-B)-\cos.(A+B)\}=\sin. A\sin. B \quad \ldots \quad (d).$$

If, in these four formulæ, $A+B=S$, and $A-B=D$, then $A=\frac{1}{2}(S+D)$, and $B=\frac{1}{2}(S-D)$; substituting these values in the four formulæ above, and multiplying both sides by 2, they become

$$\sin. S+\sin. D=2\sin.\tfrac{1}{2}(S+D)\cos.\tfrac{1}{2}(S-D) \quad \ldots \quad (e),$$

$$\sin. S-\sin. D=2\cos.\tfrac{1}{2}(S+D)\sin.\tfrac{1}{2}(S-D) \quad \ldots \quad (f),$$

$$\cos. D+\cos. S=2\cos.\tfrac{1}{2}(S+D)\cos.\tfrac{1}{2}(S-D) \quad \ldots \quad (g),$$

$$\cos. D-\cos. S=2\sin.\tfrac{1}{2}(S+D)\sin.\tfrac{1}{2}(S-D) \quad \ldots \quad (h).$$

(2.) Also, since S and D are any two arcs, they may be represented by A and B; then the above four become

$$\sin. A+\sin. B=2\sin.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B) \quad \ldots \quad (i),$$

$$\sin. A-\sin. B=2\cos.\tfrac{1}{2}(A+B)\sin.\tfrac{1}{2}(A-B) \quad \ldots \quad (k),$$

$$\cos. B+\cos. A=2\cos.\tfrac{1}{2}(A+B)\cos.\tfrac{1}{2}(A-B) \quad \ldots \quad (l),$$

$$\cos. B-\cos. A=2\sin.\tfrac{1}{2}(A+B)\sin.\tfrac{1}{2}(A-B) \quad \ldots \quad (m).$$

(3.) These four formulæ, expressed in words, prove the following important trigonometrical propositions:

(i.) The sum of the sines of two arcs is equal to twice the sine of half the sum of the arcs, multiplied into the cosine of half the difference of the arcs.

(*k*.) The difference of the sines of two arcs, is equal to twice the cosine of half the sum of the arcs, multiplied into the sine of half the difference of the arcs.

(*l*.) The sum of the cosines of two arcs is equal to twice the cosine of half the sum of the arcs, multiplied into the cosine of half the difference of the arcs.

(*m*.) The difference of the cosines of two arcs is equal to twice the sine of half the sum of the arcs, multiplied into the sine of half the difference of the arcs.

(4.) Again, resuming the expressions (A), (B), (D), and (E) in Cor. 4, Prop. XII.; namely,

$$\sin.(A+B) = \sin. A \cos. B + \cos. A \sin. B \quad \ldots \quad (A),$$

$$\sin.(A-B) = \sin. A \cos. B - \cos. A \sin. B \quad \ldots \quad (B),$$

$$\cos.(A+B) = \cos. A \cos. B - \sin. A \sin. B \quad \ldots \quad (E),$$

$$\cos.(A-B) = \cos. A \cos. B + \sin. A \sin. B \quad \ldots \quad (D).$$

In each of these four expressions, let $B = A$; then

(A) becomes $\sin. 2A = 2 \sin. A \cos. A$ ... ... (*n*),

(B) " $\sin. 0 = 0$ ... ... (*p*),

(E) " $\cos. 2A = \cos.^2 A - \sin.^2 A$ ... ... (*q*),

and (D) " $\cos. 0 = \cos.^2 A + \sin.^2 A = 1$ (Trig. 10), (*r*).

(5.) In the expressions in (2), let $B = 0$, and remember that it has just been proved that $\sin. 0 = 0$, and $\cos. 0 = 1$; then each of the expressions (*i*) and (*k*) give (*n*) as above; while

(*l*) gives $1 + \cos. A = 2 \cos.^2 \frac{1}{2}A$ ... ... (*s*),

and (*m*) " $1 - \cos. A = 2 \sin.^2 \frac{1}{2}A$ ... ... (*t*).

(6.) Again, divide the expressions in (2) by one another, and they give

$$\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}{2 \cos. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}$$

$$= \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (u);$$

by dividing both numerator and denominator of the second side by $2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)$; but it was proved in (Prop. V. Trig.), that if A and B be two angles of a triangle,

and a and b the sides opposite to them, that $a : b :: \sin. A : \sin. B$, therefore (Book V. Prop. G)

$$a + b : a - b :: \sin. A + \sin. B : \sin. A - \sin. B,$$

or
$$\frac{a+b}{a-b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B};$$

therefore
$$\frac{a+b}{a-b} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)},$$

which is the rule for finding the angles of a triangle when there are given two sides and the included angle; see Trig. Prop. VI.

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}{2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}$$
$$= \tan. \tfrac{1}{2}(A+B) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (v),$$

$$\frac{\sin. A + \sin. B}{\cos. B - \cos. A} = \frac{2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}{2 \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}$$
$$= \cot. \tfrac{1}{2}(A-B) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (w),$$

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{2 \cos. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}{2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}$$
$$= \tan. \tfrac{1}{2}(A-B) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (x),$$

$$\frac{\sin. A - \sin. B}{\cos. B - \cos. A} = \frac{2 \cos. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}{2 \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}$$
$$= \cot. \tfrac{1}{2}(A+B) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (y),$$

$$\frac{\cos. B + \cos. A}{\cos. B - \cos. A} = \frac{2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)}{2 \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)}$$
$$= \frac{\cot. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)} = \frac{\cot. \frac{1}{2}(A-B)}{\tan. \frac{1}{2}(A+B)} \quad \ldots \quad (z).$$

In the last three expressions if $B = 0$, they become, remembering that by (p) and (r) $\sin. 0 = 0$, and $\cos. 0 = 1$,

$$\frac{\sin. A}{1 + \cos. A} = \tan. \tfrac{1}{2}A \quad \ldots \quad \ldots \quad \text{from } (x),$$

$$\frac{\sin. A}{1 - \cos. A} = \cot. \tfrac{1}{2}A \quad \ldots \quad \ldots \quad \ldots \quad (y),$$

$$\frac{1 + \cos. A}{1 - \cos. A} = \frac{\cot. \frac{1}{2}A}{\tan. \frac{1}{2}A} = \cot.^2 \tfrac{1}{2}A \quad \ldots \quad \ldots \quad (z).$$

Or, by inverting,

$$\frac{1 - \cos. A}{1 + \cos. A} = \frac{\tan. \frac{1}{2}A}{\cot. \frac{1}{2}A} = \tan.^2 \tfrac{1}{2}A.$$

(7.) To find expressions for the tangent and cotangent of the sum and difference of two arcs.

Divide the expression (A) by (E) in (4), and we have

$$\frac{\sin. (A + B)}{\cos. (A + B)} = \frac{\sin. A \cos. B + \cos. A \sin. B}{\cos. A \cos. B - \sin. A \sin. B},$$

or $$\tan. (A + B) = \frac{\tan. A + \tan. B}{1 - \tan. A \tan. B};$$

by dividing numerator and denominator of the second side by cos. A cos. B, and remembering that $\frac{\sin.}{\cos.} = \tan.$ Also, let B = A, and it becomes

$$\tan. 2A = \frac{2 \tan. A}{1 - \tan.^2 A}.$$

Again, dividing (B) by (D), gives

$$\frac{\sin. (A - B)}{\cos. (A - B)} = \frac{\sin. A \cos. B - \cos. A \sin. B}{\cos. A \cos. B + \sin. A \sin. B},$$

or $$\tan. (A - B) = \frac{\tan. A - \tan. B}{1 + \tan. A \tan. B}.$$

Next divide (E) by (A), and we have

$$\frac{\cos. (A + B)}{\sin. (A + B)} = \frac{\cos. A \cos. B - \sin. A \sin. B}{\sin. A \cos. B + \cos. A \sin. B},$$

or $$\cot. (A + B) = \frac{\cot. A \cot. B - 1}{\cot. B + \cot. A};$$

by dividing numerator and denominator of the second side by sin. A sin. B.

Let now B = A, and the last expression becomes

$$\cot. 2A = \frac{\cot.^2 A - 1}{2 \cot. A}.$$

Lastly, divide (D) by (B), and we have

$$\frac{\cos. (A - B)}{\sin. (A - B)} = \frac{\cos. A \cos. B + \sin. A \sin. B}{\sin. A \cos. B - \cos. A \sin. B},$$

or $$\cot.(A-B)=\frac{\cot. A\cot. B+1}{\cot. B-\cot. A}.$$

(8.) To find expressions for the secant and cosecant of the sum and difference of two arcs.

$$\sec.(A+B)=\frac{1}{\cos.(A+B)}$$

$$=\frac{1}{\cos. A\cos. B-\sin. A\sin. B}$$

$$=\frac{\sec. A\sec. B}{1-\tan. A\tan. B}$$

$$=\frac{\sec. A\sec. B}{1-\sqrt{(\sec.^2A-1)(\sec.^2B-1)}}.$$

Let $B=A$, then $\sec. 2A=\dfrac{\sec.^2A}{2-\sec.^2A}$.

Again, $$\sec.(A-B)=\frac{1}{\cos.(A-B)}$$

$$=\frac{1}{\cos. A\cos. B+\sin. A\sin. B}$$

$$=\frac{\sec. A\sec. B}{1+\tan. A\tan. B}$$

$$=\frac{\sec. A\sec. B}{1+\sqrt{(\sec.^2A-1)(\sec.^2B-1)}}.$$

Next, $$\text{cosec.}(A+B)=\frac{1}{\sin.(A+B)}$$

$$=\frac{1}{\sin. A\cos. B+\cos. A\sin. B}$$

$$=\frac{\text{cosec.} A\,\text{cosec.} B}{\cot. B+\cot. A}$$

$$=\frac{\text{cosec.} A\,\text{cosec.} B}{\sqrt{(\text{cosec.}^2A-1)}+\sqrt{(\text{cosec.}^2B-1)}}.$$

Let $B=A$, then $\text{cosec.} 2A=\dfrac{\text{cosec.}^2A}{2\sqrt{(\text{cosec.}^2A-1)}}$.

$$\text{Lastly, cosec. }(A-B)=\frac{1}{\sin.(A-B)}$$

$$=\frac{1}{\sin. A\cos. B-\cos. A\sin. B}$$

$$=\frac{\text{cosec. A cosec. B}}{\cot. B-\cot. A}$$

$$=\frac{\text{cosec. A cosec. B}}{\sqrt{(\text{cosec.}^2 B-1)}-\sqrt{(\text{cosec.}^2 A-1)}}.$$

(9.) Since by Art. 4 (n) sin. 2A = 2 sin. A cos. A, it follows that $\sin(A+B)=2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)$, and that $\sin.(A-B)=2\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)$; if sin. (A + B) and sin. (A − B) be each divided by the expressions (i), (k), (l), and (m) in (Art. 2), there results the eight following interesting theorems:

$$\frac{\sin.(A+B)}{\sin. A+\sin. B}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}$$

$$=\frac{\cos.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A-B)}.$$

$$\frac{\sin.(A+B)}{\sin. A-\sin. B}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}{2\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)}$$

$$=\frac{\sin.\frac{1}{2}(A+B)}{\sin.\frac{1}{2}(A-B)}.$$

$$\frac{\sin.(A+B)}{\cos. A+\cos. B}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}{2\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}$$

$$=\frac{\sin.\frac{1}{2}(A+B)}{\cos.\frac{1}{2}(A-B)}.$$

$$\frac{\sin.(A+B)}{\cos. B-\cos. A}=\frac{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A+B)}{2\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)}$$

$$=\frac{\cos.\frac{1}{2}(A+B)}{\sin.\frac{1}{2}(A-B)}.$$

$$\frac{\sin.(A-B)}{\sin. A+\sin. B}=\frac{2\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)}{2\sin.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}$$

$$=\frac{\sin.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)}.$$

$$\frac{\sin.(A-B)}{\sin. A-\sin. B}=\frac{2\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)}{2\cos.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)}$$
$$=\frac{\cos.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)}.$$

$$\frac{\sin.(A-B)}{\cos. A+\cos. B}=\frac{2\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)}{2\cos.\frac{1}{2}(A+B)\cos.\frac{1}{2}(A-B)}$$
$$=\frac{\sin.\frac{1}{2}(A-B)}{\cos.\frac{1}{2}(A+B)}.$$

$$\frac{\sin.(A-B)}{\cos. B-\cos. A}=\frac{2\sin.\frac{1}{2}(A-B)\cos.\frac{1}{2}(A-B)}{2\sin.\frac{1}{2}(A+B)\sin.\frac{1}{2}(A-B)}$$
$$=\frac{\cos.\frac{1}{2}(A-B)}{\sin.\frac{1}{2}(A+B)}.$$

(10.) The preceding expressions have only been proved for arcs in the first quadrant, or for angles not greater than 90°; but they are true for arcs of any magnitude, if the *signs* of the quantities be attended to, it being a convention in trigonometry that if a vertical and a horizontal diameter be drawn, all the trigonometrical lines that are perpendicular to the horizontal diameter are *plus* when drawn above it, and *minus* when drawn below it; while those that are perpendicular to the vertical diameter are *plus*, when drawn to the right, and *minus* when drawn to the left. Hence it is evident that the sine of an arc is *plus* in the first and second quadrants, and *minus* in the third and fourth; while the cosine is *plus* in the first and fourth quadrants, and *minus* in the second and third.

The other trigonometrical lines or ratios are all functions of the sine and cosine, that is, can be expressed in terms of them, and therefore their signs can be found from these expressions by the ordinary rules of Algebra; thus tan. A $=\frac{\sin. A}{\cos. A}$, and cot. A $=\frac{\cos. A}{\sin. A}$, hence the tangent and cotangent of any arc will be *plus*, when the sine and cosine of the same arc have the same sign, that is, in the first and third quadrants; and *minus* when the sine and cosine have opposite signs, that is, in the second and fourth quadrants. Lastly, the sec. A $=\frac{1}{\cos. A}$, and cosec. A $=\frac{1}{\sin. A}$; hence the secant will have the same sign as the cosine, and the cosecant will have the same sign as the sine; hence the secant will be *plus* in the first and fourth quadrants, and *minus* in the second and third; but the cosecant will be *plus* in the first and second quadrants, and *minus* in the third and fourth.

(11.) To find the numerical values of the trigonometrical ratios of 45°, 30°, 60°, 18°, 15°, and 75°.

In a right-angled triangle when one of the acute angles is 45°, the other is also 45°, therefore the sides opposite these angles are also equal (Euc. I. 6), and hence the sine and cosine of 45° are equal; but (Trig. 10),

$$\sin.^2 45^\circ + \cos.^2 45^\circ = 1, \text{ or } 2\sin.^2 45^\circ = 1,$$

therefore $\sin.^2 45^\circ = \frac{1}{2}$;

hence $\sin. 45^\circ = \frac{1}{\sqrt{2}}$, and $\cos. 45^\circ = \frac{1}{\sqrt{2}}$.

Again $\frac{\sin. 45^\circ}{\cos. 45^\circ} = \tan. 45^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = 1,$

but $\cot. 45^\circ = \frac{1}{\tan. 45^\circ} = \frac{1}{1} = 1;$

$$\sec. 45^\circ = \frac{1}{\cos. 45^\circ} = \sqrt{2}, \text{ and } \operatorname{cosec.} 45^\circ = \frac{1}{\sin. 45^\circ} = \sqrt{2}.$$

By (Prop. 12, Cor. 4, c) $\sin. 2A = 2\sin. A \cos. A$,

therefore $\sin. 60^\circ = 2\sin. 30^\circ \cos. 30^\circ$, but $\cos. 30^\circ = \sin. 60^\circ$,

hence $\sin. 60^\circ = 2\sin. 30^\circ \sin. 60^\circ$;

and dividing both sides by $2\sin. 60^\circ$, $\sin. 30^\circ = \frac{1}{2}$,

but $\cos. 30^\circ = \sqrt{1 - \sin.^2 30^\circ} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$.

And since the sine of an angle is equal to the cosine of its compliment,

$$\cos. 60^\circ = \tfrac{1}{2}, \text{ and } \sin. 60^\circ = \frac{\sqrt{3}}{2},$$

from which it is easily found that

$$\tan. 30^\circ = \frac{1}{\sqrt{3}}, \cot. 30^\circ = \sqrt{3}, \sec. 30^\circ = \frac{2}{\sqrt{3}},$$

and $\operatorname{cosec.} 30^\circ = 2$;

$$\tan. 60^\circ = \sqrt{3}, \cot. 60^\circ = \frac{1}{\sqrt{3}}, \sec. 60^\circ = 2,$$

and $\operatorname{cosec.} 60^\circ = \frac{2}{\sqrt{3}}$.

Since it has been shewn in the corollaries to Proposition Tenth of Book Fourth, that the angle subtended at the centre by the greater segment of the radius divided medially is 36°, half the greater segment of the radius divided medially is the sine of 18°; now if x represent the greater segment, and radius be 1, by Proposition Eleventh of Book Second, the following equation is obtained:

$$x^2 = 1 - x, \text{ or } x^2 + x = 1;$$

whence $x = \dfrac{\sqrt{5}-1}{2}$, and therefore $\sin. 18^\circ = \dfrac{\sqrt{5}-1}{4}$,

and therefore $\cos. 18^\circ = \sqrt{1 - \sin.^2 18^\circ}$

$$= \sqrt{\frac{10+2\sqrt{5}}{16}} = \sqrt{\frac{5+\sqrt{5}}{8}} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}};$$

hence also $\sin. 72^\circ = \dfrac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$, and $\cos. 72^\circ = \dfrac{\sqrt{5}-1}{4}$.

Lastly, $\sin. 15^\circ = \sin. (60^\circ - 45^\circ)$

$$= \sin. 60^\circ \cos. 45^\circ - \cos. 60^\circ \sin. 45^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos. 75^\circ,$$

and $\cos. 15^\circ = \cos. (60^\circ - 45^\circ)$

$$= \cos. 60^\circ \cos. 45^\circ + \sin. 60^\circ \sin. 45^\circ$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin. 75^\circ.$$

The first ten of the following exercises in Trigonometry, may be solved by means of the values found above, and the rules formerly established, without the aid of trigonometrical tables.

EXERCISES.

1. The hypotenuse of a right-angled triangle is 200, and one of its acute angles is 30°; find the two sides.

Ans. 100 and $100\sqrt{3} = 173{\cdot}205$.

2. The base of a right-angled triangle is 80, and the vertical angle is 30°; find the hypotenuse and the perpendicular.

Ans. Hyp. = 160, perp. = $80\sqrt{3}$.

3. The base of a right-angled triangle is 30, and each of its acute angles is 45°; find the hypotenuse and perpendicular.

Ans. Hyp. $= 30\sqrt{2}$, and the perp. $= 30$.

4. The base of a triangle is 150, and the angles at the base are 60° and 75°; find the third angle and the remaining sides.

Ans. The third angle $= 45°$, and the sides are $75\sqrt{6}$ and $75(\sqrt{3} + 1)$.

5. The vertical angle of an isosceles triangle is 30°, and the base is 200; find the length of the equal sides.

Ans. The sides $= 100(\sqrt{6} + \sqrt{2})$.

6. In the triangle of last question find the length of the perpendicular from the vertex on the base, and the perpendicular from the extremity of the base on one of the equal sides.

Ans. $100(2 + \sqrt{3})$ and $50(\sqrt{6} + \sqrt{2})$.

7. The three sides of a triangle are 7, $7\sqrt{3}$, and 14; find the three angles. . . . . . . Ans. 30°, 60°, and 90.

8. One angle of a triangle is 45°, and the side opposite to it is 80, another side is $40\sqrt{6}$; find the remaining angles, and the third side. . Ans. 60°, 75°, and the third side $40(\sqrt{3} + 1)$.

9. A tower stands on a horizontal plane, the angle of elevation of its top above the plane at a certain point was 45°, and 120 feet further from it the same angle was 30°; find the height of the tower. . . . . . . Ans. $60(\sqrt{3} + 1)$.

10. One side of a triangle is 96 feet, and the angle at the base adjacent to it is 75°; find the perpendicular from the vertex on the base and the distance of the foot of the perpendicular from the angle.

Ans. Perp. $= 24(\sqrt{6} + \sqrt{2})$, and dist. $= 24(\sqrt{6} - \sqrt{2})$.

11. If in any triangle a perpendicular be drawn from the vertex upon the base, the segments of the base have the same ratio as the tangents of the parts into which the vertical angle is divided.

12. The base of a triangle is to the sum of its two sides, as the cosine of half the sum of the angles at the base to the cosine of half their difference.

13. The base of a triangle is to the difference of its sides, as the sine of half the sum of the angles at the base to the sine of half their difference.

14. The base of a triangle is to the difference of its segments, as

the sine of the vertical angle to the sine of the difference of the angles at the base.

15. Half the perimeter of a triangle is to its excess above the base, as the cotangent of half either of the angles at the base to the tangent of half the other angle.

16. The excess of half the perimeter of a triangle above the less side is to its excess above the greater, as the tangent of half the greater angle at the base to the tangent of half the less.

17. In a right-angled triangle, radius is to the sine of double one of the acute angles, as the square of half the hypotenuse to the area of the triangle.

18. Radius is to the tangent of half the vertical angle of a triangle, as the rectangle under half the perimeter and its excess above the base to the area of a triangle.

APPENDIX.

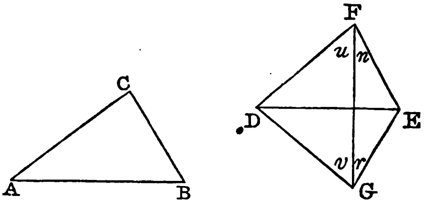
PROPOSITION (A). THEOREM.

[To be substituted for I. 7 and 8 of Elements.]

If two triangles have three sides of the one respectively e to the three sides of the other, they are equal in every respect

Given ABC, DEF two triangles, having the side A DF, CB = FE, and AB = DE; *to prove that* they are e way equal.

(*Const.*) For let the triangle ABC be applied to DEF, so t the point A may coincide with D, and the line AB with then shall the point B coincide with E; and let the s



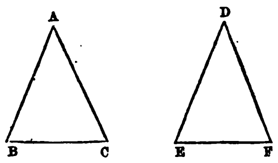
AC, CB take respectively the positions DG and GE, and j FG.

(*Dem.*) Then because FD and DG are each equal to AC, is equal to DG, therefore (I. 5) the angle DGF is equal to angle DFG; and because FE and EG are each equal to FE is equal to EG, therefore the angle EGF is equal to angle EFD. But the angle DGF was proved equal to angle DFG, therefore the whole angle DGE is equal to whole angle DFE. Again, the angle DGE is the a ACB in a different position, therefore AC and CB and contained angle ACB, are equal to DF and FE and contained angle DFE; hence (I. 4) the triangles are eq in every respect.

PROPOSITION XXVI.

If this proposition had been transposed, and placed after Proposition 32, its demonstration might have been given in a more simple manner, and as no use is made of it until it is applied in demonstrating the 34th Proposition, the logical sequence would not be affected by such transposition. It might then be demonstrated as follows:

If two triangles have two angles of the one equal to two angles of the other, and a side of the one equal to a corresponding side of the other; the triangles are equal in every respect.



Given two triangles ABC and DEF, which have the two angles ABC and ACB of the one, respectively equal to DEF and DFE of the other, and consequently (I. 32, Cor. 3) the third angle BAC of the one equal to the third angle EDF of the other; also let BC be *given* equal to EF; it is required *to prove that* the triangles are equal in every respect.

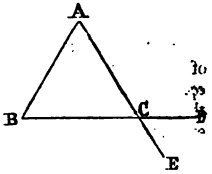
(Dem.) Conceive the triangle ABC to be applied to the triangle DEF, so that the point B may be on E, and the side BC on EF, the point C will coincide with the point F, because BC is equal to EF; and since the angle at B is equal to the angle at E, the side BA will fall on ED, also since the angle at C is equal to the angle at F, the side CA will fall on FD; therefore the point A will coincide with the point D, and since B coincides with E, and A with D, the side AB is equal to DE; and since the point C coincides with F, and A with D, the side AC is equal to the side DF; the triangles thus coinciding are equal in every respect, and have their sides equal that are opposite to the equal angles, and the area of the one is equal to the area of the other.

PROPOSITION XXXII.

There are some interesting ways in which this proposition may be demonstrated, which are not generally known; two of these are here given, they are as follow, and depend on the following principle, which may be demonstrated by the 13th Proposition. If a line be turned round a point or points in it till its ends be reversed, it has been turned through *two right angles.*

Let ABC be any triangle, and suppose a rod longer than any of its sides to be laid along BA, suppose it to be turned round the point B till it coincide with BC, it will then have changed its

direction by the angle ABC. Again, suppose it to be turned round the point C, from the position BC to AC, it will then have changed its direction by the angle BCA. Lastly, let it be turned round the point A, from the position CA to BA, it will then have changed its direction by the angle CAB. It has thus changed its position by the three angles of the triangle; but after the first change, the end which was at first above A would be at D; after the



second change, the same end would evidently be at E; and after the third change of position, it would be below the point B in the straight line AB; and hence its position has just been reversed by describing the three angles of the triangle. Therefore the three angles of any triangle are together equal to two right angles.

Second:

Suppose a person standing at B, and looking in the direction BA, turns round till he be looking in the direction BC, he will have turned through the angle ABC; let him now walk along to C without turning, he will now be looking to D in BC produced, let him now turn round till his face be in the direction E in AC produced, he will now have turned round through the angle DCE, which is equal to the angle BCA (I. 15); let him now walk backwards to A, still looking in the direction AE, and then at A turn round through the angle CAB, he will now be looking in the direction AB instead of BA; having now turned through the three angles of the triangle ABC his position is reversed, and he has therefore turned through two right angles; hence all the angles of a triangle are equal to *two right angles.*

PROPOSITION XXXV. *to* A *inclusive.*

In these propositions it should be observed that when the parallelograms and triangles are proved equal, they are only proved equal in area, not in any other respect. Some writers on Geometry, instead of saying that the parallelograms and triangles are equal, say they are equivalent; it might probably be better to say that they are equiareal, meaning thereby that they are equal in area; but this word not being in common use, the ordinary phrase has been retained, but the teacher should be careful to explain that the word *equal* in these propositions does not mean equal in any respect except that of *area.*

PROPOSITION XLIII.

In this proposition, when it is proved that the complements of the parallelograms about the diagonal of any parallelogram are *equal*, it is only proved that they are equal in area, but it may in this case always be proved that these complements are also equiangular.

PROPOSITION XLV.

There are two other ways in which this construction may be effected more simply than that given in Euclid's Elements; referring to the figure given in Euclid—first, let AB be produced beyond B, and through C draw CG parallel to DB meeting AB produced in G, and join DG; the triangle DBG is equal to the triangle DCB, for they are on the same base DB and between the same parallels DB and CG (I. 37), to each of these equals add the triangle DAB, and the triangle DAG is equal to the whole figure ABCD. Bisect AG in F, and at the point F make the angle GFH equal to the given angle E, through D draw DHK parallel to AG, and through G draw GK parallel to FH, then it is plain, by Proposition 42, that the parallelogram HFGK is equal to the triangle DAG, and is therefore also equal to the given rectilineal figure ABCD, and it has by construction the angle HFG equal to the given angle E, which was required to be done.

Second:

If the diagonal DB be bisected in F, and at the point F an angle be made equal to E, and the line forming this angle be produced both ways to meet parallels to DB through C and A, and a line be also drawn through B parallel to the line which makes the given angle at F, and produced to meet the parallels through C and A, a parallelogram will be formed, which can easily be proved, by Proposition 42, to be equal to the given quadrilateral figure, and to have an angle equal to the given rectilineal angle.

SECOND BOOK.

The first eight propositions of this book are all true both algebraically and arithmetically, if for *rectangle*, *product* be substituted; for *line*, *number*; and for *square* the *second power* of the letter or number, which is also called the square of the letter or number.

The reason of this is, that a line can be represented by the number of times that it contains a given line, and by omitting the measuring unit, the line becomes a number. Again, since the rectangle contained by two lines is the area of a right-angled parallelogram, which has these lines for two adjacent sides; and this area is represented by the number of squares on the unit of measure which it contains, and this area is always the product of the length multiplied into the breadth, into the unit of measure which is one, the area is the product of the length into the breadth. Also the square on a line is the area of the figure, which is expressed numerically by the length of the side multiplied into itself and again into the unit of measure, which is taken as one; the area of a square is therefore represented by the length of its side expressed numerically multiplied into itself or the second power of the length of the side; and this is the reason why the second power of a number is called its square. These relations being established, the propositions above referred to may all be proved algebraiacally and numerically, which should make them still more interesting to the student, and give him a more comprehensive idea of their nature and use. In proving the propositions thus, the following axiom will be assumed:

AXIOM.

If equals be multiplied by equals, the products are equal.

PROPOSITION I. THEOREM.

Referring to the diagram in the Second Book, assume the length of the line A $= a$, BC $= b$, BD $= c$, DE $= d$, and EC $= e$, then $b = c + d + e$; and multiplying each of these equals by a, $ab = ac + ad + ae$, but $ab =$ the area of the rectangle BH, $ac =$ the area of the rectangle BK, $ad =$ the area of the rectangle DL, and $ae =$ the area of the rectangle EH; and therefore the fact

that $ab = ac + ad + ae$ proves the truth of the proposition. Again, assume $a = 6$, $b = 12$, $c = 5$, $d = 3$, and consequently $e = 4$; then

$$12 \times 6 = 5 \times 6 + 3 \times 6 + 4 \times 6, \text{ or } 72 = 30 + 18 + 24 = 72.$$

COR.—If there be two numbers, one of which is divided into any number of parts, the product of the two numbers is equal to the sum of the products of the undivided into the several parts of the divided number.

PROPOSITION II. THEOREM.

Let $AC = a$, and $CB = b$, then $AB = a + b$; and the rectangle

$$AB \cdot AC = (a + b)a,$$

and the rectangle $$AB \cdot BC = (a + b)b;$$

$$\therefore AB \cdot AC + AB \cdot BC = (a + b)(a + b) = (a + b)^2 = AB^2.$$

Next, let $a = 7$, and $b = 5$, then $a + b = 12$, and

$$12 \times 7 + 12 \times 5 = 84 + 60 = 144 = 12^2.$$

PROPOSITION III. THEOREM.

Let $AC = a$, and $CB = b$, then $AB = a + b$, and hence the rectangle

$$AB \cdot BC = (a + b)b = ab + b^2;$$

but this proves the proposition for $ab =$ the area of the figure AD, and $b^2 =$ the area of CE, and it is plain that $(a + b)b =$ the area of AE.

PROPOSITION IV. THEOREM.

Let $AC = a$, and $CB = b$, then $AB = (a + b)$, and hence

$$AB^2 = (a + b)^2 = (a + b)a + (a + b)b = a^2 + ab + ab + b^2$$
$$= a^2 + b^2 + 2ab = AC^2 + CB^2 + 2AC \cdot CB.$$

It will easily be seen that $HF = a^2$, $CK = b^2$, while AC and CE are each $= ab$, and therefore together $= 2ab$.

PROPOSITION V. THEOREM.

Let $AC = CB = a$, $CD = b$, then $AD = a + b$, and $DB = a - b$, and hence the rectangle

$$AD \cdot DB + CD^2 = (a + b)(a - b) + b^2 = (a + b)a - (a + b)b$$
$$+ b^2 = a^2 + ab - ab - b^2 + b^2 = a^2 = CB^2.$$

PROPOSITION VI. THEOREM.

Let AC = CB = a, and BD = b, then AB = $2a + b$, and CD = $a + b$; whence

$$\text{AD}\cdot\text{DB} + \text{CB}^2 = (2a + b)b + a^2 = 2ab + b^2 + a^2$$
$$= a^2 + ab + ab + b^2 = (a + b)a + (a + b)b = (a + b)(a + b)$$
$$= (a + b)^2 = \text{CD}^2.$$

PROPOSITION VII. THEOREM.

Let AC = a, and CB = b, then AB = $a + b$, and hence twice the rectangle

$$\text{AB}\cdot\text{BC} + \text{AC}^2 = 2(a + b)b + a^2 = a^2 + 2ab + 2b^2$$
$$= a^2 + ab + ab + b^2 + b^2 = (a + b)a + (a + b)b + b^2$$
$$= (a + b)(a + b) + b^2 = (a + b)^2 + b^2 = \text{AB}^2 + \text{CB}^2.$$

PROPOSITION VIII. THEOREM.

Let AC = a, CB = BD = b, then AB = $a + b$, and AD = $a + 2b$, and hence four times the rectangle

$$\text{AB}\cdot\text{BC} + \text{AC}^2 = 4(a + b)b + a^2 = a^2 + 4ab + 4b^2$$
$$= a^2 + 2ab + 2ab + 4b^2 = (a + 2b)a + (a + 2b)2b$$
$$= (a + 2b)(a + 2b) = (a + 2b)^2 = \text{AD}^2.$$

PROPOSITIONS IX. AND X. THEOREMS.

These two theorems may be demonstrated algebraically by the same symbols, in each let AC = a, and CD = b; then AD = $a + b$, and in IX. BD = $a - b$, and in X. BD = $b - a$; but $(a - b)^2 = (b - a)^2$, each being $= a^2 - 2ab + b^2 = (a \sim b)^2 = \text{BD}^2$.

(*Dem.*) $\text{AD}^2 + \text{DB}^2 = (a + b)^2 + (a \sim b)^2 = a^2 + 2ab + b^2 + a^2 - 2ab + b^2 = 2a^2 + 2b^2 = 2\text{AC}^2 + 2\text{CD}^2$.

PROPOSITION XI. PROBLEM.

This proposition cannot be demonstrated numerically, for though a line can be divided geometrically, so that the rectangle contained by the whole and the less part may be exactly equal to the square of the greater, it is impossible to divide a number into two parts, so that the product of the whole number and the less part shall be exactly equal to the square of the greater part; if we solve the problem algebraically by taking the whole line as one, and its

greater segment as x, then the less segment is $(1-x)$, and the following quadratic equation is obtained, $x^2 = 1 - x$, or $x^2 + x = 1$, and therefore $x = \frac{\sqrt{5}-1}{2}$. But the square root of 5 is an interminate mixed decimal, and therefore diminishing it by 1, and dividing the remainder by 2, gives also an interminate decimal. By means of the corollary, however, a number of interesting approximations may be made to the numerical solution; for if the whole line be first considered to be 2, and its segments each 1, then the line made up of the whole and its greater segment (Cor.) will be 3, the greater segment 2, and the less segment 1; again, repeating the same operation with 3 and its greater segment, the number 5 is obtained, its greater segment being 3, and its less segment 2; and so on to any extent, by which the following interesting approximations are obtained:

$2 \times 1 = 2$ $1 \times 1 = 1$	the product is greater than the square, by the whole square.
$3 \times 1 = 3$ $2 \times 2 = 4$	the product is less than the square, by a fourth part of the square.
$5 \times 2 = 10$ $3 \times 3 = 9$	the product is greater than the square, by a ninth part of the square.
$8 \times 3 = 24$ $5 \times 5 = 25$	the product is less than the square, by a twenty-fifth part of the square.
$13 \times 5 = 65$ $8 \times 8 = 64$	the product is greater than the square, by a sixty-fourth part of the square.
$21 \times 8 = 168$ $13 \times 13 = 169$	the product is less than the square, by a hundred and sixty-ninth part of the square.
$34 \times 13 = 442$ $21 \times 21 = 441$	the product is greater than the square, by $\frac{1}{441}$th part of the square.
$55 \times 21 = 1155$ $34 \times 34 = 1156$	the product is less than the square, by $\frac{1}{1156}$th part of the square.

It is evident, from the above, that the product is alternately greater and less than the square; and that it differs from the square by a less part of the square, the greater the number of parts into which the line is divided. When the line was divided into two parts, the product was double of the square; but when it is divided into 55 parts, and the greater taken 34 and the less 21, the product is then less than the square by $\frac{1}{1156}$th part of the square. In the same manner, the difference may be made much less, by taking a greater number of divisions in the line; the next number of divisions would be 89, and then the product would exceed the square by $\frac{1}{3025}$th part of the square.

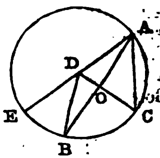
THIRD BOOK.

PROPOSITION C. THEOREM.

[To be substituted for III. 20 of the Elements.]

In the demonstration of the second part of this proposition, the following is assumed as an axiom: 'If there be four quantities such that the first is double of the second, and the third also double of the fourth, the difference of the first and third will also be double of the difference of the second and fourth.' This not having been stated as an axiom, nor being sufficiently self-evident, the following demonstration may be substituted.

Let D, the centre of the circle, be without the angle BAC; join AD, and produce it to E, and let AB, DC intersect each other in O. The exterior angle BOC of the triangle BOD is equal to the angles ODB and DBO; but the angle DBO is equal to the angle DAO, because DB is equal to DA, therefore the angle BOC is equal to the angles ODB and DAO. Again, the exterior angle BOC of the triangle AOC is equal to the angles OCA and OAC; and since DA is equal to DC, the angle OCA is equal to the angle DAC, which is equal to the two angles CAO and DAO, therefore the angle BOC is equal to twice the angle OAC and the angle DAO. But the same angle was proved to be equal to the angles ODB and DAO, therefore the angles ODB and DAO are equal to twice the angle CAO and the angle DAO; take away the common angle DAO, and there remains the angle ODB equal to twice the angle CAO; that is, the angle at the centre is double of the angle at the circumference.

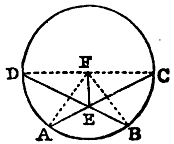


PROPOSITION (D). THEOREM.

[To be substituted for III. 35 of the Elements.]

If two chords in a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other.

Let the two chords AC, BD, within the circle ABCD, cut one another in the point E; the rectangle contained by AE, EC is equal to the rectangle contained by DE · EB.



Join FA, FB, FC, and FD, F being the centre. Then in the isosceles triangle AFC, the square on AF is equal to the rectangle AE · EC and the square on FE (II. D).

Again, in the isosceles triangle DFB, the square on DF, that is, of AF, is equal to the rectangle DE · EB and the square on FE (II. D).

Therefore the rectangle AE · EC and the square on FE is equal to the rectangle DE · EB and the square on FE; take the square on FE from each, and there remains the rectangle AE · EC equal to the rectangle DE · EB.

PROPOSITION XXXV. THEOREM.

In the same manner, as stated in the Appendix to the Second Book, this proposition may be applied practically to find the segments of lines drawn in a circle and cutting each other, for when three of the segments are given, the fourth may be found. The following exercises are appended to shew the way in which it may be applied. The letters refer to the figures in the text of Book Third.

(1.) The chord of an arc is 24, and its height 8, find the diameter of the circle.

Referring to figure (2) $AC = 24$, and $BE = 8$; therefore AE and EC are each 12; hence, since $BE \cdot ED = AE \cdot EC$,

$$8 \times DE = 12 \times 12 = 144;$$

hence $$DE = 18.$$

But $BE + DE =$ the diameter $= 8 + 18 = 26$.

(2.) A segment of a wheel being found, its chord was measured, and found to be 18 inches, and the height was found to be 3 inches; what was the diameter of the wheel when entire?

Ans. 30 inches.

(3.) The diameter of a circle is 36 inches, and the height of an arc is 12 inches; find the chord of the arc.

Ans. $24\sqrt{2}$ inches.

(4.) Two chords cut one another within a circle, the segments of the one are 8 and 9, one of the segments of the other is 6; find the other segment, and which of them cuts off the greater segment from the circle.

Ans. 12; and that whose segments are 6 and 12 cuts off the greater segment, for this chord is 18, and the other 17, it is therefore nearer to the centre (III. 15).

PROPOSITION XXXVI.

(1.) If from a point without a circle, whose diameter is 12, a straight line is drawn through the centre, whose whole length is 16; what is the length of a tangent to the circle drawn from the same point?

Here since the whole line passing through the centre is 16, and the diameter is 12, the part without the circle is 4; hence, from the proposition, the square of the tangent is equal to 16×4, and therefore the tangent $= \sqrt{16 \times 4} = \sqrt{64} = 8$.

(2.) If the length of a tangent to a circle drawn from a point without it be 12, and the length of a line passing through the centre and drawn from the same point be 18, find the distance of the point from the circle and the diameter of the circle.

Referring to figure (1), by the proposition the rectangle $AD \cdot DC$ is equal to the square on BD, or $AD \times DC = BD^2$, but $AD = 18$, and $BD = 12$; $\therefore 18 \times DC = 12^2$, hence $DC = \frac{144}{18} = 8$. Again, $AD = AC + CD$, or $AC = AD - CD = 18 - 8 = 10$; the distance of the point without the circle is therefore 8, and the diameter of the circle is 10.

(3.) Find the diameter of the earth, having given that from the top of a hill one mile high, an observer may see to the distance of 89 miles. . . . . Ans. 7920.

(4.) Having given that the earth is a sphere whose diameter is 7920 miles, find how far above its surface the eye of an observer must be placed to see to a distance of 10 miles.

Ans. $66\frac{2}{3}$ feet nearly.

(5.) How far must a point be from the circumference of a circle whose diameter is 12, so that the tangent drawn from it to the circle may be 8? . . . . Ans. 4.

(6.) Find, in terms of the diameter, the distance of a point without a circle, from which a tangent being drawn, it shall be equal to the diameter.

Let the distance sought be $= x$, the diameter $= d$, and the tangent also $= d$, then $(d + x)x = d^2$, or $x^2 + dx = d^2$; $\therefore x = \frac{d}{2}(\sqrt{5} - 1)$; or $x =$ the greater segment of the diameter divided medially.

THE END.

Edinburgh:
Printed by W. and R. Chambers.

Miscellaneous Publications.

Chambers's Journal of Popular Literature, Science, and Arts. Issued in Weekly Numbers, at 1½d.; in Monthly Parts, at 7d. and 8½d.; and in Half-yearly Volumes, cloth, at 4s. 6d.—Volume 12 completed.

Chambers's Edinburgh Journal. SECOND SERIES. In 20 Vols., price L.3, 7s. in sheets, or L.4, 10s. bound in cloth. Each volume separately, 4s. 6d.

Chambers's Encyclopædia, a Dictionary of Universal Knowledge for the People; in Weekly Numbers, at 1½d.; in Monthly Parts, at 7d.; and in Yearly Volumes, cloth, at 9s.—Volume 1 completed.

The Gallery of Nature: a Pictorial and Descriptive Tour through Creation. Illustrated with Wood-cuts and Steel Engravings. A New and Revised Edition. By the Rev. T. MILNER, M.A., F.R.G.S., &c. In One Volume, royal 8vo, cloth lettered, price 18s.; also half-bound, calf, at 21s.

Pictorial Chronicle of the Revolt in India, and of the Transactions in China in 1856-7, &c. Illustrated with Maps, Plans, and Wood-cuts. Royal 8vo. In One handsome Volume, cloth lettered, price 14s.

Chambers's History of the Russian War. Illustrated with numerous *Wood Engravings, Coloured Prints,* and *Maps.* In One handsome Volume, cloth lettered, price 13s.

Life and Works of Robert Burns. Edited by ROBERT CHAMBERS. *Library Edition.* Elegantly printed in demy 8vo, with Steel-plate Portrait of Burns and Wood Engravings. In 4 handsome Volumes, cloth, price 20s.
Also in Two Volumes, cloth lettered, price 20s.

Shakspere's Works. CHAMBERS's Improved Issue of KNIGHT's CABINET EDITION. With SUPPLEMENTARY NOTES and Wood Engravings. In Twelve Volumes, cloth, price 18s.; cloth gilt, price 24s.; or paper covers, price 12s.

The Pictorial Bible; being the OLD and NEW TESTAMENTS, according to the Authorised Version; with NOTES by DR KITTO. Illustrated with Steel Engravings, Wood-cuts, and Maps. In Four Volumes, price £2, 8s. handsomely bound in cloth; morocco, or calf extra, gilt edges, £4, 4s.

The Pictorial History of England, from the Earliest Times down to 1815. A NEW and IMPROVED ISSUE, revised under the care of Messrs CHAMBERS. Illustrated with upwards of 1600 Wood Engravings. In 6 Vols., royal 8vo, price £4, 10s. cloth lettered.

The History of the Peace. By HARRIET MARTINEAU. New and Illustrated Edition. Revised by the Author. In One handsome Volume, royal 8vo, price 13s. cloth lettered.

Chambers's Repository of Instructive and Amusing Tracts. In Twelve Volumes, fancy boards, at 1s. each; in Six Volumes, cloth, at 2s. each; or in cloth gilt, at 2s. 6d. each. Each Volume is illustrated with Wood-cuts, has a neatly engraved Title-page, and is complete in itself.

Chambers's Pocket Miscellany. Illustrated with Frontispieces, in 12 Volumes, cloth, price 1s. 6d. each; in cloth gilt, at 2s. each; or in 24 Volumes, price 6d. each, paper covers.

Chambers's Papers for the People. In Twelve Volumes, fancy boards, at 1s. 6d. each; or in cloth, at 3s. each; also in 96 Numbers, at 1½d. each.

Tales for Travellers. Selected from CHAMBERS's PAPERS FOR THE PEOPLE. In Two Volumes, cloth, with Frontispieces, price 2s. 6d. each.

Chambers's Historical & Literary Celebrities: being Biographical Sketches selected from CHAMBERS's PAPERS for the PEOPLE. Crown 8vo, cloth gilt, price 3s. 6d.

Chambers's Miscellany of Useful and Entertaining Tracts. In Twenty Volumes, fancy boards, at 1s. each; in Ten Volumes, cloth, at 2s. each; or in cloth gilt, at 2s. 6d. each. Illustrated with Wood Engravings.

Chambers's Library for Young People. This series, which embraces Moral and Religious Tales, History, and Poetry, is completed in Twenty Volumes, cloth, illustrated with Frontispieces, price 1s. each; *or in 10 vols. (2 vols. in one) cloth gilt, price 2s. 6d. each.*

mbers's People's Editions. A series of Original, Translated, and Selected Works, printed in double columns, 8vo, and, from their cheapness, adapted for the use of the less opulent classes of the community.

Miscellaneous Publications

CONTINUED.

Chambers's Information for the People. *New and Greatly Improved Edition, embellished by many new Wood Engravings in the best style.* Two Vols., royal 8vo, price 16*s.* cloth lettered; also half-bound, calf, at 20*s.*

Chambers's Cyclopædia of English Literature. *New Edition.* A Critical and Biographical History of English Writers in all departments of Literature; illustrated by specimens of their Writings. Two Volumes, price 16*s.* cloth lettered; also half-bound, calf, at 20*s.*

Chambers's Instructive and Entertaining Library. A series of Original and Selected Works in different departments of Literature, suitable for popular Entertainment and Instruction.—Twenty-four Volumes cloth, at prices varying from 2*s.* 6*d.* to 4*s.* 6*d.*

Chambers's Atlas for the People. *New and Improved Edition.* Royal 4to, cloth lettered, price 15*s.* This Atlas consists of 32 *Quarto* and 4 *Folio Maps*, beautifully printed in Colours; with a *Descriptive Introduction* and copious *Consulting Index.*

Chambers's Parlour Atlas. Royal 8vo, cloth lettered, price 15*s.* This Atlas consists of the same Maps, &c., as the ATLAS for the PEOPLE.

Chambers's Minor Atlas. Royal 4to, cloth lettered, price 5*s.* This Atlas consists of 15 Quarto Maps, beautifully printed in colours.

Chambers's Miniature Library. In 13 Vols., 32mo, cloth gilt, price 6*s.*; single copies, price 6*d.* each.

Selections from Chambers's Repository and Miscellany.

TALES for ROAD and RAIL.	In 5 vols.
SELECT POETRY.	" 1 vol.
HISTORY and ADVENTURE.	" 2 vols.
TRAVELS and SKETCHES of SCENERY.	" 1 vol.
ENTERTAINING BIOGRAPHY.	" 3 vols.

Each volume complete in itself, price 2*s.* cloth, or 2*s.* 6*d.* cloth gilt.

Commercial Tables; consisting of Reckoning, Interest, Annuity, Money, Weights, Measures, and other Tables; and forming a convenient Manual for the Warehouse and Counting-room. Price 3*s.* strongly half-bound.

The Youth's Companion and Counsellor. Edited by W. CHAMBERS. Frontispiece and Vignette Title. Price 5*s.* square 16mo, cloth gilt.

Things as They are in America; being the NARRATIVE of a TOUR in BRITISH AMERICA and the UNITED STATES. *Second Edition*, with HINTS to EMIGRANTS. By W. CHAMBERS. Price 6*s.* post 8vo, cloth lettered.

American Slavery and Colour. Illustrated with a MAP. By W. CHAMBERS, author of 'Things as they are in America.' Price 4*s.* 6*d.* post 8vo, cloth.

Improved Dwelling-Houses for the Humbler & other Classes in Cities. Based on the Scottish Dwelling-House System. *With Plans.* By W. CHAMBERS. Price 1*s.* 6*d.*

Peebles and its Neighbourhood; with a RUN on PEEBLES RAILWAY. By W. CHAMBERS. Illustrated with Wood-cuts. Price 1*s.* sewed, or 1*s.* 6*d.* cloth.

Works of Robert Chambers. In Seven post 8vo volumes, embellished with Vignette Engravings, cloth, 4*s.* each.

Tracings of Iceland and the Faröe Islands. By ROBERT CHAMBERS. Illustrated with Wood Engravings. Being an Account of a Visit to these Islands in the Summer of 1855. In small 8vo, sewed, price 1*s.*

Domestic Annals of Scotland, from the Reformation to the Revolution. *Second Edition.* Edited by R. CHAMBERS. In 2 Vols. demy 8vo, price 24*s.*

Edinburgh Papers. By ROBERT CHAMBERS. In Demy 8vo sewed
Ancient Domestic Architecture of
before the Siege of 1573. Price
Merchants and Merchandise in Old
Testimony; its Posture in the Sci
Romantic Scottish Ballads; their

Cookery and Domestic Economy, for
neatly bound in cloth, price 1*s.* 6*d.*